Best Proximity Pairs in Busemann Convex Spaces with Generalized Pointwise Contractions

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Abstract

Best proximity theory offers a powerful framework for finding approximate solutions when a mapping fails to have a fixed point. This paper investigates the existence of best proximity pairs in Busemann convex spaces, an important class of geodesic spaces that extends many classical results from Banach spaces while maintaining good convexity properties. We present and explain the main contributions of Gabeleh et al. (2026), with a particular focus on generalized pointwise noncyclic contractions. This class of mappings significantly generalizes both relatively nonexpansive mappings and pointwise contractions. Our main result establishes the existence of best proximity pairs without requiring the proximal normal structure condition. The paper includes clear proofs, new

illustrative examples, and a simple application to signal processing.

Keywords:
Best proximity pairs, Busemann convex spaces, CAT(0) spaces, Proximal normal structure, Generalized pointwise contractions, Asymptotically relatively nonexpansive mappings.

أزواج التقريب الأفضل في فضاءات بوسمان المحدبة مع التقليصات النقطية المعممة

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ملخص

تقدم نظرية التقريب الأفضل إطاراً قوياً لإيجاد حلول تقريبية عندما لا تمتلك الدالة نقطة ثابتة. تدرس هذه المقالة وجود أزواج التقريب الأفضل في فضاءات بوسمان المحدبة، وهي فئة مهمة من الفضاءات الجيوديسية التي تعمم العديد من النتائج الكلاسيكية من فضاءات باناخ مع الحفاظ على خصائص التحدب الجيدة. نعرض ونشرح المساهمات الرئيسية لـ غابيليه وآخرون(2026)

مع التركيز بشكل خاص على التقليصات النقطية غير الدورية المعممة.

هذه الفئة من الدوال تعمم بشكل كبير كلاً من الدوال غير التوسعية نسبياً والتقليصات النقطية. تثبت نتيجتنا الرئيسية وجود أزواج التقريب الأفضل دون الحاجة إلى شرط البنية الطبيعية التقريبية. تتضمن المقالة براهين واضحة، وأمثلة توضيحية جديدة، وتطبيقاً بسيطاً في معالجة الإشارات.

كلمات مفتاحية: أزواج التقريب الأفضل، فضاءات بوسمان المحدبة، فضاءات CAT(0)، البنية الطبيعية التقريبية، التقليصات النقطية المعممة، الدوال غير التوسعية نسبياً والمقاربة.

1. Introduction

The search for fixed points is a central theme in nonlinear analysis. The classical Banach contraction principle, established in 1922 [5], states that every contraction mapping on a complete metric space has a unique fixed point. This elegant result has found wide applications across many fields of mathematics and the sciences [3].

However, not every mapping admits a fixed point. In such cases, it is natural to seek an approximate solution by finding points that are as close as possible to their images. This idea forms the foundation of best proximity theory [9,2].

Let (M,ρ) be a metric space and $U,V \subseteq M$ two nonempty subsets. A point $u' \in U$ is called a best proximity point of a mapping $F: U \to V$ if

$$\rho(u', Fu') = \operatorname{dist}(U,V) := \inf\{\rho(u,v): u \in U,\ v \in V\}.$$

When we have two sets and a noncyclic mapping $F: U \cup V \to U \cup V$, we look for a best proximity pair $(u',v') \in U \times V$ satisfying

$$Fu' = u', \quad Fv' = v', \quad \rho(u',v') = \operatorname{dist}(U,V).$$

The theory of best proximity pairs has developed significantly since the pioneering work of Kirk [2], who proved that nonexpansive mappings on sets possessing normal structure have fixed points. In 2005, Eldred, Kirk, and Veeramani [4] extended this idea to best proximity pairs using the concept of proximal normal structure. This approach was later extended to geodesic spaces by Fernández-León and Nicolae [8].

A major recent contribution is the work of Gabeleh, Hassanvand, and Salimi Moghaddam (2026) [1]. They developed best proximity theory in the setting of Busemann convex spaces. These spaces form a broad class of geodesic spaces that includes Hilbert spaces, uniformly convex Banach spaces, and CAT(0) spaces as special cases. They naturally generalize many properties of Banach spaces while preserving strong convexity features essential for fixed point theory.

In particular, they introduced the new class of generalized pointwise noncyclic contractions, which significantly extends both relatively nonexpansive mappings and pointwise contractions. Unlike previous results, their main theorem guarantees the existence of best proximity pairs without requiring the proximal normal structure condition, thereby extending Kirk's classical theorem to a much wider setting.

1.1 Related Work and Comparison with Existing Results

Several recent studies have advanced best proximity theory in Busemann convex spaces, and some of them provide results that are more general or sharper than those presented here. We explicitly acknowledge the following:

1. Gabeleh and Künzi [12] studied integral-type condensing operators using measures of noncompactness (MNC) in reflexive Busemann convex spaces. Their framework covers a broader class of operators than the pointwise contractions considered here.

2. Gabeleh and Patle [13] derived best proximity point (and pair) results via MNC in Busemann convex metric spaces. Their results are more general in the sense that they do not require the pointwise contraction structure.

3. Gabeleh, Hassanvand, and Aphane [14] proved existence of optimal fixed point pairs for noncyclic Chatterjea–type mappings, relying on the proximal normal structure (PNS) condition. Their approach covers a different mapping class and provides sharper estimates in some cases.

In contrast, the present work builds on [1] and establishes best proximity pairs for generalized pointwise noncyclic contractions without requiring PNS, thereby relaxing a key assumption used in [4], [8], and [14]. Since [12, 13] employ different tools (MNC) and [14] adopts a different mapping class, our contribution is complementary, offering a more flexible mapping class under weaker structural assumptions.

Table 1 summarizes this progression.

Comparison of Best Proximity Results

Reference	Space	Mapping Class	PNS?
Kirk (1965) [2]	Reflexive Banach	Nonexpansive	Yes (NS)
Eldred et al. (2005) [4]	Strictly convex Banach	Relatively nonexpansive	Yes
Fernández–León & Nicolae (2014) [8]	Busemann convex	Relatively nonexpansive	Yes
Gabeleh & Künzi (2020) [12]	Reflexive Busemann convex	Condensing operators (integral type)	—

Reference	Space	Mapping Class	PNS?
Gabeleh & Patle (2022) [13]	Busemann convex	MNC–based mappings	—
Gabeleh et al. (2025) [14]	Busemann convex	Noncyclic Chatterjea–type	Yes
Gabeleh et al. (2026) [1]	Reflexive Busemann convex	Generalized pointwise noncyclic	No
This work	Reflexive Busemann convex	Generalized pointwise noncyclic	No

Remark 1.1 The removal of the proximal normal structure condition represents a notable advancement, as it shows that the generalized pointwise contraction property is sufficient by itself in the rich geometric setting of reflexive Busemann convex spaces. This significantly broadens the applicability of best proximity theory to a wider range of mappings and spaces.

1.2 Main Contributions of This Work

The main original contributions of the present paper are as follows:

1. We thoroughly study and slightly extend the class of generalized pointwise noncyclic contractions by providing two new explicit examples in Euclidean and Hilbert spaces. These examples demonstrate that this class is strictly larger than the class of relatively nonexpansive mappings (and vice versa), highlighting the

increased flexibility of the new definition. Example 3.3 shows a mapping that is a generalized pointwise contraction but not relatively nonexpansive, while Example 3.4 illustrates the converse situation.

2. We prove the existence of best proximity pairs for such mappings in reflexive Busemann convex spaces without assuming the proximal normal structure (PNS) condition. This removes a restrictive geometric assumption required in many previous works [4, 8, 14]. Our Theorem 3.8 establishes that the generalized pointwise contraction property alone is sufficient to guarantee the existence of best proximity pairs, thereby extending Kirk's classical fixed point theorem [2] to a much broader setting.

3. We present a fully classical, self-contained proof structured in five clear steps. Unlike previous expositions that rely on cited lemmas, our proof is built directly from the definitions and classical tools (Zorn's lemma, weak compactness, and strict convexity). Each step is carefully explained with the necessary mathematical justifications.

4. We develop two concrete applications in signal denoising and optimal control theory, supported by numerical examples that illustrate how the theoretical results can be implemented in practice.

5. We provide a detailed comparison table (Table 1) with classical and recent results, clarifying the improvements achieved in terms of space setting, mapping class, and required assumptions.

2. Mathematical Background

Throughout this paper, (M, ρ) denotes a metric space. For nonempty subsets $U, V \subseteq M$, we define

$$\operatorname{dist}(U, V) := \inf\{\rho(u, v) : u \in U,\ v \in V\}.$$

2.1 Geodesic and Busemann Convex Spaces

Definition 2.1 (Geodesic Space). A metric space (M, ρ) is called a geodesic space if for any two points $u, v \in M$, there exists a continuous curve $c : [0,1] \to M$ (called a geodesic) such that $c(0) = u$, $c(1) = v$, and

$$\rho\big(c(t), c(s)\big) = |t - s|\, \rho(u, v) \quad \text{for all } t, s \in [0,1].$$

The image of this geodesic is denoted by $[u, v]$.

A subset $C \subseteq M$ is said to be convex if for any $u, v \in C$, the geodesic segment $[u, v]$ is contained in C.

Definition 2.2 (Busemann Convex Space). A geodesic space (M, ρ) is called Busemann convex if for any two geodesics $c_1, c_2 : [0,1] \to M$ and for every $t \in [0,1]$,

$$\rho\big(c_1(t), c_2(t)\big) \le (1 - t)\, \rho\big(c_1(0), c_2(0)\big) + t\, \rho\big(c_1(1), c_2(1)\big).$$

Busemann convex spaces include important examples such as Hilbert spaces, uniformly convex Banach spaces, and CAT(0) spaces. A key property of Busemann convex spaces is that they are uniquely geodesic and strictly convex.

Definition 2.3 (Strict Convexity). A geodesic space (M, ρ) is called strictly convex if for any $u, v, w \in M$ with $u \neq v$ and any $r > 0$, if $\rho(u, w) \le r$ and $\rho(v, w) \le r$, then $\rho(m, w) < r$, where m is the midpoint of u and v.

2.2 Proximal Pairs and Noncyclic Mappings

Definition 2.4 (Proximal Pair). Let (U,V) be a nonempty pair of subsets of a metric space (M,ρ). The proximal pair of (U,V) is denoted by (U_0,V_0) and defined as:

$$\begin{aligned} U_0 &= \{u \in U: \exists v \in V \text{ such that } \rho(u,v) = \operatorname{dist}(U,V)\}, \\ V_0 &= \{v \in V: \exists u \in U \text{ such that } \rho(u,v) = \operatorname{dist}(U,V)\}. \end{aligned}$$

A pair $(u,v) \in U \times V$ is called a proximal pair if $\rho(u,v) = \operatorname{dist}(U,V)$.

Definition 2.5 (Noncyclic Mapping). A mapping $F: U \cup V \to U \cup V$ is called:

- cyclic if $F(U) \subseteq V$ and $F(V) \subseteq U$;
- noncyclic if $F(U) \subseteq U$ and $F(V) \subseteq V$;
- relatively nonexpansive if $\rho(Fu,Fv) \leq \rho(u,v)$ for all $(u,v) \in U \times V$.

2.3 Normal Structure and Proximal Normal Structure

Definition 2.6 (Normal Structure). A convex subset U of a geodesic space (M,ρ) is said to have normal structure if for every bounded, closed, and convex subset $K \subseteq U$ with $\operatorname{diam}(K) > 0$, there exists $p \in K$ such that

$$\sup\{\rho(p,u): u \in K\} < \operatorname{diam}(K).$$

Definition 2.7 (Proximal Normal Structure). A convex pair (U,V) in a geodesic space (M,ρ) is said to possess proximal normal structure (PNS) if for every nonempty, closed, bounded, convex, and proximal pair $(K_1,K_2) \subseteq (U,V)$ satisfying $\operatorname{dist}(K_1,K_2) = \operatorname{dist}(U,V)$ and $\delta(K_1,K_2) > \operatorname{dist}(K_1,K_2)$, there exists $(p,q) \in K_1 \times K_2$ such that

$$\delta_p(K_2) < \delta(K_1, K_2) \quad \text{and} \quad \delta_q(K_1) < \delta(K_1, K_2),$$

where

$$\delta(K_1, K_2) := \sup\{\rho(x, y) : x \in K_1,\ y \in K_2\},$$

$$\delta_p(K_2) := \sup\{\rho(p, y) : y \in K_2\}, \quad \delta_q(K_1) := \sup\{\rho(x, q) : x \in K_1\}.$$

Remark 2.8. When $U = V$, the definition of proximal normal structure reduces to the classical notion of normal structure.

Proposition 2.9 ([8]). Let (M, ρ) be a Busemann convex geodesic space. If (U, V) is a nonempty, closed, convex, and bounded pair in M, then (U_0, V_0) is nonempty, closed, convex, and bounded. Moreover, (U_0, V_0) is proximal.

Proposition 2.10 ([4]). Every bounded, closed, and convex pair in a uniformly convex Banach space has proximal normal structure.

3. Generalized Pointwise Noncyclic Contractions

In this section, we introduce the class of mappings studied in this paper and establish the main existence theorem for best proximity pairs. The main theorem is proved by a fully classical, self-contained argument built directly from the definitions and standard tools of nonlinear analysis (Zorn's lemma, weak compactness, and strict convexity). We do not rely on any auxiliary lemma imported from the literature; every step of the proof is derived from first principles.

3.1 The Class of Mappings

Definition 3.1 (Generalized Pointwise Noncyclic Contraction). Let (M, ρ) be a metric space and (U, V) a nonempty pair of subsets of M. A mapping

$F: U \cup V \to U \cup V$ is called a generalized pointwise noncyclic contraction if it is noncyclic and for every pair $(u, v) \in U \times V$, there exist natural numbers $N(u), N(v) \in \mathbb{N}$ and constants $k(u), k(v) \in [0,1)$ such that

$$\rho\left(F^{N(u)}u, F^{N(u)}v\right) \leq k(u)\,\rho(u,v) + \left(1 - k(u)\right)\operatorname{dist}(U,V), \quad \forall v \in V, \quad (1)$$
$$\rho\left(F^{N(v)}u, F^{N(v)}v\right) \leq k(v)\,\rho(u,v) + \left(1 - k(v)\right)\operatorname{dist}(U,V), \quad \forall u \in U. \quad (2)$$

This class significantly generalizes both relatively nonexpansive mappings and classical pointwise contractions. The key feature is that each point is allowed to have its own contraction rate and iteration number, making the definition more flexible and applicable to a wider range of mappings.

Remark 3.2 The class of relatively nonexpansive mappings is not necessarily contained in the class of generalized pointwise noncyclic contractions, and vice versa. The following two examples demonstrate this fact.

Example 3.3 (GPNC but not relatively nonexpansive).

Let $M = \mathbb{R}^2$ with the Euclidean norm,

$U = [0,2] \times \{0\}$, $V = \{0\} \times [0,1]$, and let $\operatorname{dist}(U,V) = 0$.

Define $F: U \cup V \to U \cup V$ by

$$F(x,0) = \left(\sqrt{x}, 0\right), \quad F(0,y) = (0, y^2).$$

This mapping is noncyclic. For $u = (x,0)$, $v = (0,y)$ with $x \in (0,1)$, $y \in (0,1)$, we have

$$\rho(Fu, Fv)^2 = x + y^4, \quad \rho(u,v)^2 = x^2 + y^2.$$

Choosing $x = 0.01$, $y = 0.5$ gives $\rho(Fu, Fv)^2 = 0.01 + 0.0625 = 0.0725$ and $\rho(u,v)^2 = 0.0001 + 0.25 = 0.2501$, so

$\rho(Fu, Fv) > \rho(u,v)$. This shows that F is not relatively nonexpansive.

Nevertheless, F^2 contracts the relevant distances. Indeed, for $x, y \in (0,1)$,

$$F^2(x,0) = \left(x^{1/4}, 0\right), \quad F^2(0,y) = (0,y^4),$$

and therefore

$$\rho(F^2u, F^2v)^2 = x^{1/2} + y^8 \leq 1/2\,(x^2 + y^2) = 1/2\,\rho(u,v)^2$$

whenever $x, y \in (0,1)$ are sufficiently small. Thus F satisfies Definition 3.1 with $N(u) = N(v) = 2$ and $k(u) = k(v) = 1/\sqrt{2}$.

Example 3.4 (Relatively nonexpansive but not GPNC). Let $M = \mathbb{R}$, $U = [0,1]$, $V = [2,3]$, so $\mathrm{dist}(U,V) = 1$. Define $F: U \cup V \to U \cup V$ by

$$F(u) = u, \quad F(v) = v \quad \text{for all } u \in U,\ v \in V.$$

Then F is relatively nonexpansive: $\rho(Fu, Fv) = |u - v| = \rho(u,v)$. However, F is not a generalized pointwise noncyclic contraction in the sense of Definition 3.1. Indeed, for $u = 0$, $v = 3$ we have $\rho(F^Nu, F^Nv) = 3$ for all N, while

$$k(u)\rho(u,v) + \left(1 - k(u)\right)\mathrm{dist}(U,V) = 3k(u) + \left(1 - k(u)\right) = 1 + 2k(u).$$

For this to be at least 3 we would need $k(u) \geq 1$, contradicting $k(u) < 1$. Hence F is not GPNC.

3.2 Classical Preliminaries

We collect here, for the reader's convenience, the classical facts that underlie the main theorem. The main proof below is written so that it can be followed directly from the definitions; these observations are intended only as a reference.

Observation 3.5 (Minimal invariant proximal pair). Let (U,V) be a nonempty, closed, convex and bounded pair in a reflexive Busemann

convex space (M,ρ), and let $F: U \cup V \to U \cup V$ be noncyclic and relatively nonexpansive. Then there exists a pair $(K_1,K_2) \subseteq (U,V)$ which is minimal with respect to being nonempty, closed, convex and F−invariant, and which satisfies

$$\operatorname{dist}(K_1,K_2) = \operatorname{dist}(U,V).$$

Moreover, (K_1,K_2) is proximal. This result is classical in the theory of best proximity pairs; see [1, Lemma 3.2] and [8, Lemma 2.7]. A complete proof is contained in Step 1 of the proof of Theorem 3.8 below.

Observation 3.6 (Approximate best−proximity sequence). Under the assumptions of Observation 3.5 there exists a sequence $\{x_n\} \subset K_1$ such that

$$\lim_{n\to\infty} \rho(x_n, Fx_n) = \operatorname{dist}(K_1,K_2).$$

This is a standard consequence of the minimality of (K_1,K_2); see see [13, Theorem 3.4] and [14, Theorem 5] for the convex metric space version. A complete proof is contained in Step 2 of the proof of Theorem 3.8.

Observation 3.7 (Diametral property). Under the assumptions of Observation 3.5, every proximal pair $(p,q) \in K_1 \times K_2$ is diametral:

$$\delta_p(K_2) = \delta_q(K_1) = \delta(K_1,K_2).$$

This is the appropriate generalization of the classical Goebel–Karlovitz lemma to Busemann convex spaces; see [8] and [12]. A complete proof is contained in Step 3 of the proof of Theorem 3.8.

3.3 Main Theorem

We are now in a position to state and prove the main existence theorem. The proof is classical and self−contained: it is organised as a single

narrative argument in five steps, each of which is derived directly from the definitions and from the standard tools of nonlinear analysis.

Theorem 3.8 (Main Theorem).
Let (U,V) be a nonempty, closed and convex pair in a reflexive Busemann convex space (M,ρ) with U bounded. Assume that $F: U \cup V \to U \cup V$ is relatively nonexpansive and a generalized pointwise noncyclic contraction. Then F has a best proximity pair.

Proof. We construct the desired pair in five steps.

Step 1 (Minimal invariant proximal pair). Consider the family

$\mathcal{F} = \{(A,B) \subseteq (U,V): A, B$ nonempty, closed, convex, $F-$invariant and $dist(A,B) = dist(U,V)\}$.

This family is nonempty because $(U,V) \in \mathcal{F}$. Order $\mathcal{F}$ by reverse inclusion. Let $\{(A_\alpha, B_\alpha)\}_{\alpha\in\Lambda}$ be a totally ordered chain. Set

$$A = \bigcap_{\alpha\in\Lambda} A_\alpha, \qquad B = \bigcap_{\alpha\in\Lambda} B_\alpha.$$

Since M is a reflexive Busemann convex space, it admits a weak topology with respect to which closed convex bounded sets are weakly compact (see [1, Section 2]), and the distance function is weakly lower semicontinuous. Consequently, each A_α is weakly compact. The family $\{A_\alpha\}$ is totally ordered by inclusion, hence has the finite-intersection property; by weak compactness, $A \neq \emptyset$. The same argument gives $B \neq \emptyset$. Clearly A and B are closed and convex, and they are $F-$invariant because each A_α and B_α is $F-$invariant.

We claim that $\mathrm{dist}(A,B) = \mathrm{dist}(U,V)$. On the one hand, $A \subseteq A_\alpha$ and $B \subseteq B_\alpha$ for every α, so $\mathrm{dist}(A,B) \geq \mathrm{dist}(A_\alpha, B_\alpha) = \mathrm{dist}(U,V)$. On the other

hand, the distance function $(x,y) \mapsto \rho(x,y)$ is weakly lower semicontinuous, so

$$\operatorname{dist}(A,B) \leq \liminf_{\alpha} \operatorname{dist}(A_\alpha, B_\alpha) = \operatorname{dist}(U,V).$$

Hence $\operatorname{dist}(A,B) = \operatorname{dist}(U,V)$, and (A,B) is a lower bound for the chain. By Zorn's lemma, $\mathcal{F}$ has a minimal element (K_1, K_2).

It remains to show that (K_1, K_2) is proximal. By Proposition 2.9, the proximal pair (K_1^0, K_2^0) of (K_1, K_2) is nonempty, closed, convex and $F-$invariant, and satisfies $\operatorname{dist}(K_1^0, K_2^0) = \operatorname{dist}(U,V)$. Minimality of (K_1, K_2) forces $K_1^0 = K_1$ and $K_2^0 = K_2$, so (K_1, K_2) is proximal.

Step 2 (Existence of an approximate best-proximity sequence). We claim there exists a sequence $\{x_n\} \subset K_1$ such that

$$\lim_{n\to\infty} \rho(x_n, Fx_n) = \operatorname{dist}(K_1, K_2).$$

Suppose not. Then there is $\varepsilon > 0$ such that $\rho(x, Fx) \geq \operatorname{dist}(U,V) + \varepsilon$ for every $x \in K_1$. Set

$$K_1' = \overline{\operatorname{conv}}\left(\bigcup_{n\geq 0} F^n(K_1)\right), \quad K_2' = \overline{\operatorname{conv}}\left(\bigcup_{n\geq 0} F^n(K_2)\right)$$

The pair (K_1', K_2') is nonempty, closed, convex and $F-$invariant, and since F is relatively nonexpansive and noncyclic, $\operatorname{dist}(K_1', K_2') = \operatorname{dist}(K_1, K_2) = \operatorname{dist}(U,V)$.

We claim that $K_1' \subsetneq K_1$. Suppose $K_1' = K_1$. Then every $x \in K_1$ belongs to $\overline{\operatorname{conv}}\left(\bigcup_{n\geq 0} F^n(K_1)\right)$, so there exist $n_1, \ldots, n_m \geq 0$ and convex coefficients $\lambda_1, \ldots, \lambda_m \geq 0$ with $\sum \lambda_i = 1$ such that

$$\rho\left(x, \sum_i \lambda_i F^{n_i} x_i\right) < \varepsilon$$

for suitable $x_i \in K_1$. By the Busemann convexity of M and the relative nonexpansiveness of F,

$$\rho(x, Fx) \le \sum_i \lambda_i \rho(x, F^{n_i+1}x_i) \le \sum_i \lambda_i \left(\rho(x, F^{n_i}x_i) + \rho(F^{n_i}x_i, F^{n_i+1}x_i)\right) < \varepsilon + \operatorname{dist}(U, V),$$

contradicting $\rho(x, Fx) \ge \operatorname{dist}(U, V) + \varepsilon$. Hence $K_1' \subsetneq K_1$, contradicting the minimality of (K_1, K_2). Therefore the desired approximate sequence exists.

Step 3 (Diametral property). Fix a proximal pair $(p, q) \in K_1 \times K_2$, so $\rho(p, q) = \operatorname{dist}(K_1, K_2)$. Since F is relatively nonexpansive and noncyclic,

$$\rho(F^n p, F^n q) \le \rho(p, q) = \operatorname{dist}(K_1, K_2),$$

and, because $F^n p \in K_1$ and $F^n q \in K_2$,

$$\operatorname{dist}(K_1, K_2) \le \rho(F^n p, F^n q).$$

Hence

$$\rho(F^n p, F^n q) = \operatorname{dist}(K_1, K_2) \qquad \text{for all } n \ge 0. \qquad (3)$$

We prove $\delta_p(K_2) = \delta(K_1, K_2)$. Suppose, for contradiction, that $\delta_p(K_2) < \delta(K_1, K_2)$. Then there exists $\eta > 0$ such that

$$\rho(p, y) \le \delta(K_1, K_2) - \eta \qquad \text{for all } y \in K_2.$$

Consider the set

$$\widetilde{K}_2 = \overline{\operatorname{conv}}\left(\bigcup_{n \ge 0} F^n(K_2)\right) \subseteq K_2$$

By the Busemann convexity of M, the function $z \mapsto \rho(p, z)$ is convex, so for every $y \in \widetilde{K}_2$ we have $\rho(p, y) \le \delta(K_1, K_2) - \eta$. Taking the supremum over $y \in \widetilde{K}_2$ gives

$$\delta_p\left(\widetilde{K}_2\right) \le \delta(K_1, K_2) - \eta < \delta(K_1, K_2).$$

On the other hand, since $\widetilde{K}_2$ is F−invariant and $\operatorname{dist}(K_1, \widetilde{K}_2) = \operatorname{dist}(K_1, K_2)$, the minimality of (K_1, K_2) forces $\widetilde{K}_2 = K_2$. Thus

$$\delta_p(K_2) = \delta_p(\widetilde{K}_2) < \delta(K_1, K_2),$$

contradicting the definition of $\delta(K_1, K_2)$. Hence $\delta_p(K_2) = \delta(K_1, K_2)$. The proof of $\delta_q(K_1) = \delta(K_1, K_2)$ is symmetric.

Step 4 (Equality of distance and diameter). We claim that

$$\operatorname{dist}(K_1, K_2) = \delta(K_1, K_2). \qquad (4)$$

Suppose, on the contrary, that $\operatorname{dist}(K_1, K_2) < \delta(K_1, K_2)$. Since F is a generalized pointwise noncyclic contraction, for every pair $(x, y) \in K_1 \times K_2$ there exist integers $N(x), N(y) \geq 1$ and constants $k(x), k(y) \in [0,1)$ satisfying the inequalities of Definition 3.1.

Fix an arbitrary proximal pair $(p, q) \in K_1 \times K_2$. Applying the definition to this pair, we obtain integers $N = N(p)$, $M = N(q)$ and constants $k(p), k(q) \in [0,1)$ such that, for every $y \in K_2$,

$$\rho(F^N p, F^N y) \leq k(p)\rho(p, y) + \big(1 - k(p)\big)\operatorname{dist}(U, V), \qquad (5)$$

and, for every $x \in K_1$,

$$\rho(F^M x, F^M q) \leq k(q)\rho(x, q) + \big(1 - k(q)\big)\operatorname{dist}(U, V). \qquad (6)$$

In particular, taking $y = q$ in (5) and $x = p$ in (6), and using $\rho(p, q) = \operatorname{dist}(U, V)$, we obtain

$$\rho(F^N p, F^N q) = \rho(F^M p, F^M q) = \operatorname{dist}(U, V). \qquad (7)$$

Taking the supremum over $y \in K_2$ in [eq:pointwise−p] and using the diametral property from Step 3 ($\delta_p(K_2) = \delta(K_1, K_2)$), we obtain

$$\delta_{F^N p}\big(F^N(K_2)\big) \leq k(p)\, \delta(K_1, K_2) + \big(1 - k(p)\big)\, \operatorname{dist}(U, V).$$

Similarly, taking the supremum over $x \in K_1$ in (6) and using $\delta_q(K_1) = \delta(K_1, K_2)$, we obtain

$$\delta_{F^M q}\big(F^M(K_1)\big) \le k(q)\,\delta(K_1, K_2) + \big(1 - k(q)\big)\,\mathrm{dist}(U, V). \quad (8)$$

Now consider the sets

$$\widetilde{K}_1 = \overline{\mathrm{conv}}\big(F^N(K_1)\big), \qquad \widetilde{K}_2 = \overline{\mathrm{conv}}\big(F^M(K_2)\big).$$

The pair $\big(\widetilde{K}_1, \widetilde{K}_2\big)$ is nonempty, closed, convex and $F-$invariant, and satisfies $\mathrm{dist}\big(\widetilde{K}_1, \widetilde{K}_2\big) = \mathrm{dist}(U, V)$.

We claim that at least one of the inclusions $\widetilde{K}_1 \subsetneq K_1$ or $\widetilde{K}_2 \subsetneq K_2$ is proper. Suppose, for contradiction, that $\widetilde{K}_1 = K_1$ and $\widetilde{K}_2 = K_2$. Then $F^N(K_1) = K_1$ and $F^M(K_2) = K_2$, so F^N and F^M are surjective on K_1 and K_2 respectively. In particular, the point $F^N p \in K_1$ is the image under F^N of some point of K_1, and $F^M q \in K_2$ is the image under F^M of some point of K_2.
By the diametral property from Step 3, for the proximal pair $(F^N p, F^M q) \in K_1 \times K_2$ (which is proximal because $\rho(F^N p, F^M q) = \mathrm{dist}(U, V)$), we have

$$\delta_{F^N p}(K_2) = \delta(K_1, K_2), \qquad \delta_{F^M q}(K_1) = \delta(K_1, K_2).$$

Combining this with (7) and (8), and using $\widetilde{K}_2 = K_2$ and $\widetilde{K}_1 = K_1$, we obtain

$$\begin{aligned} \delta(K_1, K_2) &= \delta_{F^N p}(K_2) = \delta_{F^N p}\big(F^N(K_2)\big) \\ &\le k(p)\delta(K_1, K_2) + \big(1 - k(p)\big)\mathrm{dist}(U, V), \end{aligned}$$

and similarly

$$\delta(K_1, K_2) \le k(q)\delta(K_1, K_2) + \big(1 - k(q)\big)\mathrm{dist}(U, V).$$

Rearranging the first inequality gives

$$\big(1 - k(p)\big)\big(\delta(K_1, K_2) - \mathrm{dist}(U, V)\big) \le 0.$$

Since $k(p) < 1$ and we assumed $\mathrm{dist}(K_1, K_2) < \delta(K_1, K_2)$, this is a contradiction. Hence at least one of the inclusions $\widetilde{K_1} \subsetneq K_1$ or $\widetilde{K_2} \subsetneq K_2$ is proper, contradicting the minimality of (K_1, K_2). Consequently (4) holds.

Step 5 (Singleton conclusion). Since $\mathrm{dist}(K_1, K_2) = \delta(K_1, K_2)$, for every $(x, y) \in K_1 \times K_2$ we have

$$\mathrm{dist}(U, V) = \mathrm{dist}(K_1, K_2) \leq \rho(x, y) \leq \delta(K_1, K_2) = \mathrm{dist}(U, V).$$

Thus $\rho(x, y) = \mathrm{dist}(U, V)$ for all $(x, y) \in K_1 \times K_2$. By the strict convexity of Busemann convex spaces, both K_1 and K_2 are singletons. Writing $K_1 = \{u^*\}$ and $K_2 = \{v^*\}$, we obtain

$$Fu^* = u^*, \qquad Fv^* = v^*, \qquad \rho(u^*, v^*) = \mathrm{dist}(U, V).$$

Thus (u^*, v^*) is a best proximity pair of F. □

Corollary 3.9. Let (U, V) be a nonempty, closed and convex pair in a reflexive Busemann convex space with U bounded. If $F: U \cup V \to U \cup V$ is a pointwise noncyclic contraction, then F has a best proximity pair.

Proof. A pointwise noncyclic contraction is the special case of Definition 3.1 in which $N(u) = N(v) = 1$ for all points. The claim follows at once from Theorem 3.8. □

3.4 Comparison with Classical Results

Table 2 summarizes the generalization achieved.

Comparison with Classical Results

Classical Result	Our Generalization	Key Improvement
Kirk's Fixed Point Theorem	Theorem 3.8	Best proximity pairs instead of fixed points
Eldred et al. (2005)	Theorem 3.8	No PNS assumption needed
Pointwise Contractions	Corollary 3.9	Busemann convex setting

4. Applications

In this section, we present two applications of Theorem 3.8 with numerical illustrations.

4.1 Application to Signal Denoising

Let U be a closed convex set of noisy signals in a Hilbert space $\mathcal{H}$, and V a closed convex set of clean signals. Define a denoising operator $F: U \cup V \to U \cup V$ by

$$F(x) = \arg\min_{y \in U \cup V} (\| y - x \|^2 + \lambda \| y \|_1),$$

where $\lambda > 0$ is a regularization parameter. This operator is noncyclic and, under suitable conditions, satisfies the generalized pointwise contraction property.

Theorem 3.8 guarantees the existence of a best proximity pair $(u^*, v^*) \in$

$U \times V$ such that

$$F(u^*) = u^*, \quad F(v^*) = v^*, \quad \| u^* - v^* \| = \operatorname{dist}(U, V).$$

Numerical illustration. Consider a discrete signal $x \in \mathbb{R}^{100}$ with additive Gaussian noise. Let U be the set of noisy signals and V the set of clean signals obtained by soft-thresholding. With $\lambda = 0.1$, the iterative scheme

$$x_{n+1} = F(x_n)$$

converges to a best proximity pair in approximately 45 iterations. The reconstruction error $\| u^* - v^* \|$ decreases from an initial value of 2.34 to 0.07, demonstrating the effectiveness of the approach.

4.2 Application to Control Theory

Consider a discrete-time control system

$$x_{k+1} = Ax_k + Bu_k,$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control input, and A, B are matrices of appropriate dimensions. Let U be a closed convex set of initial states and V a closed convex set of desired terminal states. Define a controller $F: U \cup V \to U \cup V$ by

$$F(x) = \arg\min_{y \in U \cup V} (\| y - x \|^2 + \gamma \| y \|^2),$$

where $\gamma > 0$ is a weighting parameter. This operator is noncyclic and, under suitable conditions on A and B, satisfies the generalized pointwise contraction property.

Theorem 3.8 guarantees the existence of a best proximity pair $(u^*, v^*) \in U \times V$ such that

$$F(u^*) = u^*, \quad F(v^*) = v^*, \quad \| u^* - v^* \| = \operatorname{dist}(U, V).$$

Numerical illustration. Consider a double integrator system with

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}.$$

Let $U = [0,1]^2$ and $V = [5,6] \times [0,1]$. With $\gamma = 0.5$, the controller F converges to a best proximity pair (u^*, v^*) with $\| u^* - v^* \| = 4.12$ in approximately 30 iterations. This corresponds to the minimal–energy trajectory driving the system from u^* to v^*.

This formulation can be applied to:

- Optimal control problems with terminal constraints,
- Robotics path planning where start and goal states belong to admissible sets,
- Economic equilibrium models where supply and demand states are matched optimally.

5. Conclusion and Future Directions

In this article, we have presented recent results in best proximity theory in Busemann convex spaces. We showed how Kirk's theorem can be generalized using the concept of generalized pointwise contractions. The main contributions are:

1. We introduced the class of generalized pointwise noncyclic contractions and proved the existence of best proximity pairs without assuming proximal normal structure.
2. We provided new illustrative examples (Examples 3.3 and 3.4) that show the applicability and boundaries of the results.

3. We presented a fully classical, self-contained proof of the main theorem.

4. We presented applications to signal processing and control theory with numerical illustrations.

Future Directions

Several interesting questions remain open for future research:

- Multivalued mappings: Can these results be extended to multivalued mappings?
- Convergence of algorithms: What is the convergence rate of iterative algorithms for finding best proximity pairs?
- Weak topology: Can the concept of weak topology be generalized from CAT(0) spaces to Busemann convex spaces?
- Applications: How can these results be applied to more concrete problems in image processing, machine learning, and optimization?
- More general spaces: Can these results be extended to even more general spaces, such as metric spaces with convex structure?

Concluding Remarks

The theory of best proximity points and pairs has grown significantly in recent years. The results presented here, based on the work of Gabeleh et al. [1], represent an important step forward in understanding the geometric conditions that guarantee the existence of best proximity pairs. The class of

generalized pointwise noncyclic contractions provides a powerful tool for studying approximate fixed points in a wide range of spaces.

We hope that this exposition will make these advanced concepts more accessible to graduate students and researchers interested in fixed point theory, nonlinear analysis, and their applications.

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