Existence, Uniqueness, and Convergence of Solutions for General Monotone Inclusions in Hilbert Spaces

Boushra Rajab ABBAS
Associate professor at Latakia University
boushraabbas@latakia-univ.edu.sy

Abstract

Monotone inclusions play a central role in applied mathematics, particularly in optimization, variational inequalities, game theory, and physical modeling. This paper generalizes the Newton-like dynamics previously introduced by Abbas et al. for solving inclusions of the form $0 \in M(x) = A(x) + B(x)$ in Hilbert spaces, where A and B are maximal monotone operators, and B is also assumed to be monotone and locally Lipschitz continuous. Unlike earlier works, such as A being a subdifferential or B being a gradient, thus making the method more general. We establish existence and uniqueness of strong solutions using the resolvent operator and demonstrate both weak and strong convergence under appropriate conditions. A Lyapunov function-based analysis is employed to rigorously study the asymptotic behavior. We also explore a discrete version of the proposed dynamics, offering practical algorithms for numerical computation. Applications in mechanical systems, control theory, electrical circuits, and machine learning are discussed, showcasing the method's flexibility and effectiveness. Finally, we highlight the potential for further extensions via adaptive regularization and inertial methods.

Keywords: Monotone inclusions, Hilbert spaces, Newton–like dynamics, maximal monotone operators, forward–backward splitting, resolvent operator, convergence analysis, Levenberg–Marquardt regularization, Lyapunov functions, optimization, engineering applications.

مجلة جامعة حمص سلسلة العلوم الأساسية
المجلد 47 العدد 18 عام 2025 أ.م.د. بشرى رجب عباس

دراسة وجود ووحدانية وتقارب حلول مسائل الاحتواءات المضطردة العامة في فضاءات هيلبرت

أ.م.د. بشرى رجب عباس
أستاذ مساعد في قسم الرياضيات كلية العلوم جامعة تشرين
boushraabbas@latakia-univ.edu.sy

ملخص

تُعدّ الاحتواءات المضطردة عنصراً أساسياً في الرياضيات التطبيقية، خصوصاً في مجال الأمثليات، ومتباينات المتغيّرات، ونظرية الألعاب، ونمذجة الأنظمة الفيزيائية. في هذه المقالة، نُعمّم الأنظمة الديناميكية الشبيهة بمنظومات نيوتن، والتي قُدمت سابقًا من قِبل عباس وآخرين، لحل مسائل من الشكل:

$$0 \in M(x) = A(x) + B(x)$$

في فضاءات هيلبرت، حيث أن A, B مؤثرين مضطردين أعظميين، كما أن B مؤثر ليبشيتز.

على عكس الأعمال السابقة التي تطلبت افتراضات بنيوية—مثل كون A مؤثر تحت تفاضل لدالة محدبة و B تدرج لدالة مستمرة، نقوم هنا بإزالة هذه القيود، مما يجعل المنهج المقترح أكثر عمومية وشمولية. نُثبت وجود ووحدانية الحلول القوية باستخدام مؤثر حال(Resolvent) ، كما نستعرض شروطًا تضمن تحقق التقارب الضعيف والقوي. ويُستخدم تحليل قائم على دوال ليابونوف لدراسة السلوك اللانهائي بشكل دقيق وصارم.

فضلًا عن ذلك، ندرس النسخة المتقطعة من الأنظمة الديناميكية المقترحة، مما يوفر خوارزميات عملية للتطبيقات العددية. ونناقش تطبيقات متعددة تشمل الأنظمة الميكانيكية، ونظرية التحكم، والدوائر الكهربائية، وتعلم الآلة، بما يُبرز مرونة وكفاءة المنهج المطروح. وأخيرًا، نُشير إلى إمكانيات التعميم المستقبلية، بما في ذلك التنظيم التكيفي وطرائق القصور الذاتي.

كلمات مفتاحية: الاحتواءات المضطردة، فضاءات هيلبرت، الأنظمة الديناميكية الشبيهة بنيوتن، طريقة ليفنبريغ- ماركارد في التنظيم، تحليل ليابونوف، الأمثليات.

1. Introduction

In recent decades, monotone inclusion problems have emerged as a powerful unifying framework for a broad range of mathematical models across optimization [1], control theory [2], game theory [3], and physical sciences [4]. These problems typically formulated as

$$0 \in M(x) = A(x) + B(x)$$

where A and B are monotone operators on a real Hilbert space $H -$ encapsulate numerous structured and unstructured systems, from variational inequalities [7] and equilibrium formulations [6] to nonsmooth evolution equations.

The challenge of solving such inclusions becomes significantly more intricate when A is set−valued and nonsmooth, and when the system lacks explicit gradient structures. While classical methods such as proximal point algorithms [4] or splitting schemes (like forward−backward or Douglas–Rachford) have been extensively studied [7], their convergence can be slow or inapplicable in more general settings.

A particularly promising direction arises from dynamical system approaches [10], where time−continuous flows are designed to asymptotically solve the inclusion. In this context, Abbas et al. [5] introduced a Newton−like differential inclusion incorporating Levenberg–Marquardt−type regularization, under the assumption that

$A = \partial\varphi$ for some convex φ, and $B = \nabla\psi$ for a smooth function ψ. Their work demonstrated strong convergence properties and inspired a new generation of continuous-time algorithms.

In this work, we go significantly beyond such structural assumptions. We study a generalized Newton-like dynamical system where both A and B are allowed to be arbitrary maximal monotone operators, without requiring subdifferential or gradient representations. Our framework thus applies to a much broader class of problems, including those with nonsmooth, set-valued, or non-potential components.

Our main contributions are as follows:

- We prove existence and uniqueness of strong global solutions using the theory of resolvent operators. [11]
- We establish both weak and strong convergence results under natural conditions.
- A discrete version of the dynamic is derived, enabling efficient numerical implementation[12].
- Finally, we demonstrate the practical versatility of the proposed framework through applications in mechanical systems [13], control [2], electrical networks [14], and machine learning [15].

2. Mathematical Background

Monotone for addressing complex problems in optimization,

variational inequalities, game theory,
partial differential equations, and engineering systems. In a real Hilbert space H with inner product $\langle .,. \rangle$ and norm $\|.\|$, the aim is to find $x \in H$ such that $0 \in M(x)$, where $M = A + B$ with $A: H \to 2^H$ a maximal monotone operator and $B: H \to H$ monotone and locally Lipschitz continuous. A maximal monotone operator is monotone
i.e.

$$\langle u - v, x - y \rangle \geq 0 \ \ for\ all\ u \in A(x), v \in A(y)$$

and cannot be extended while preserving monotonicity; common examples include subdifferentials of convex functions and normal cones to convex sets. While classical Newton's method

$$x_{k+1} = x_k - \left[\acute{M}(x_k)\right]^{-1} M(x_k)$$

is effective for smooth single−valued mappings, it fails for nonsmooth or multivalued operators common in this setting. To overcome this, we employ Newton−like dynamics with Levenberg–Marquardt regularization, introducing a time−dependent parameter $\mu(t) > 0$ to stabilize the evolution and enable convergence rates close to Newton's method. Central to the analysis is Minty's resolvent transformation, where the resolvent $J_{A,\mu} = (I + \mu A)^{-1}$ is single−valued and nonexpansive, and the Yosida approximation $A_\mu = \frac{1}{\mu}\left(I - J_{A,\mu}\right)$ is Lipschitz continuous, allowing the use of Cauchy–Lipschitz theory for well−posedness.

Rewriting the inclusion

$$\dot{x}(t) + \mu(t)A\big(x(t)\big) + B\big(x(t)\big) \ni 0$$

via the change of variables

$$z(t) = x(t) + \mu(t)v(t), v(t) \in \big(x(t)\big)$$

, yields the equivalent forward–backward splitting form

$$x(t) = J_{A,\mu} z(t),$$
$$\dot{z}(t) + (\mu(t) - \dot{\mu}\ (t)\ \ )A_{\mu(t)} z(t) + \mu(t)B\left(J_{A,\mu(t)} z(t)\right) = 0$$

which requires only resolvent evaluations of A (implicit step) and explicit evaluations of B (forward step). This general formulation, which does not assume A is a subdifferential or B a gradient, extends the framework to nonsmooth and multivalued problems such as frictional contact in mechanics, control in robotics, and queue optimization in machine learning. Under the stated assumptions, there exists a unique strong global solution an absolutely continuous trajectory satisfying the inclusion almost everywhere for a given initial condition and convergence can be analyzed via Lyapunov functions. If the solution set is nonempty and $\int_0^\infty \mu(t)dt = \infty$, trajectories converge weakly to a
solution; strong convergence requires additional properties such as strong or uniform monotonicity of B or uniqueness of the minimizer. The discrete time analogue $x_{k+1} = J_{A,\mu_k}\big(x_k - \mu_k B(x_k)\big)$, with $\sum_{k=1}^{\infty} \mu_k = \infty$ and $\mu_k \to 0$, , retains these convergence properties and is well−suited for

numerical optimization and control.

3. Existence and Uniqueness

Given the general framework above, we now address the well-posedness of the Newton-like dynamics.

We consider the differential inclusion

$$\dot{x}(t) + \mu(t)A\big(x(t)\big) + B\big(x(t)\big) \ni 0, \;\; t \geq 0$$

with the initial condition

$$x(0) = x_0 \in H$$

A function $x\colon [0, \infty[\to H$ is called a strong solution if:

1. $x(t)$ is absolutely continuous on $[0, \infty[$.
2. For almost every $t \geq 0$, there exists $v(t) \in A\big(x(t)\big)$ such that

$$\dot{x}(t) + \mu(t)v(t) + B\big(x(t)\big) = 0$$

3. $x(0) = x_0$ for a given initial condition $x_0 \in H$.

We assume that

- $A\colon H \to 2^H$ a maximal monotone operator.
- $B\colon H \to H$ monotone and locally Lipschitz continuous.
- $\mu\colon [0, \infty[\to]0, \infty[$ is continuous and satisfies

$$\lim_{t \to \infty} \mu(t) > 0$$

By Minty's theorem, the resolvent

$$J_{A,\mu} = (I + \mu A)^{-1}$$

is single-valued and nonexpansive.

Using this property, the inclusion can be rewritten as the fixed-point equation

$$x(t) = J_{A,\mu}\left(x(t) - \mu(t)B\big(x(t)\big)\right)$$

Defining

$$F(t,x) = J_{A,\mu(t)}\left(x(t) - \mu(t)B\big(x(t)\big)\right) - x$$

We note that $F(t,x)$ is locally Lipschitz in x and continuous in t.

Since the trajectories remain bounded in H, F has linear growth, and by the Cauchy–Lipschitz theorem, there exists a unique strong global solution $x(t)$ for all $t \geq 0$.

4. Convergence of Trajectories

We now study the asymptotic behavior of the strong global solution $x(t)$ to the differential inclusion

$$\dot{x}(t) + \mu(t)A\big(x(t)\big) + B\big(x(t)\big) \ni 0, \quad t \geq 0$$

With $x(0) = x_0 \in H$,under the same standing assumptions stated earlier. Let

$$S = \{x \in H, 0 \in A(x) + B(x)\}$$

be the set of zeros of $M = A + B$, which is assumed to be nonempty. The analysis distinguishes between weak and strong convergence, and is based on Lyapunov−type arguments and monotonicity properties of the operators.

If The regularization parameter $\mu: [0,\infty[\to]0,\infty[$ satisfies

$\int_0^\infty \mu(t)dt = \infty$

then every trajectory $x(t)$ converges weakly in H to some $x^* \in S$ as

$t \to \infty$. To prove this, one considers the energy functional

$$E(t) = \frac{1}{2}\|x(t) - x^*\|^2$$

which is differentiable almost everywhere, and shows that

$$\acute{E}(t) \leq -\mu(t)\langle v(t) - v^*, x(t) - x^* \rangle$$

for appropriate $v(t) \in A\big(x(t)\big)$ and $v^* \in A(x^*)$ satisfying $0 = v^* + B(x^*)$. By the monotonicity of A and B, the right−hand side is nonpositive, implying that $E(t)$ is nonincreasing and bounded from below. Hence $E(t)$ has a finite limit $\|x(t) - x^*\|$ remains bounded, and the sequence of weak cluster points of $x(t)$ is nonempty. Passing to the limit in the inclusion using the demiclosedness of maximal monotone operators shows that all such cluster points belong to S, which ensures that the whole trajectory converges weakly to a point in S.

To obtain strong convergence, additional structure is needed. If B is strongly monotone with modulus $\alpha > 0$, i.e.,

$$\langle B(x) - B(y), x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in H,$$

then the solution set S reduces to a single point $\{x^*\}$, and the weak convergence result strengthens to

$$\lim_{t\to\infty}\|x(t) - x^*\| = 0$$

Other sufficient conditions for strong convergence include uniform monotonicity of BBB, or in the special case where A and $\boldsymbol{B}$ are subdifferentials of convex functions, the strict convexity of the associated potential function. Under these assumptions, the Lyapunov function $E(t)$ not only decreases to a finite limit but forces $\|x(t)-x^*\| \rightarrow 0$ as $t \rightarrow \infty$.

5. Challenges in Achieving Strong Convergence

Achieving strong convergence requires imposing stringent conditions on B, such as strong or uniform monotonicity, which are often restrictive in practical applications. Strong monotonicity

$$\langle B(x)-B(y), x-y\rangle \geq \alpha\|x-y\|^2 \quad x \geq y,$$

implies a linear growth in the monotonicity condition, which may not hold for operators arising in nonsmooth optimization or physical systems where B models nonlinear forces [5]. Uniform monotonicity, with $\langle\langle B(x)-B(y), x-y\rangle \geq \alpha\|x-y\|^2$ if $\|x-y\|$ is less restrictive but requires ϕ to be strictly increasing and positive, which still excludes many operators. Moreover, the monotonicity condition is often difficult to verify. For example, assuming a singleton $argmin\, M$ with B continuous can yield strong convergence [2], but verifying a unique solution is challenging in high−dimensional spaces. Weaker conditions, like cocoercivity or Lipschitz continuity, typically result in weaker convergence, as the Lyapunov function decay rate is insufficient [1]. These constraints limit the framework's applicability to systems lacking strong monotonicity, necessitating alternative approaches, such as inertial methods or adaptive regularization, to enhance convergence without overly restrictive assumptions [4].

6. Applications and Extensions

6.1 Discrete Version of the Dynamical System

The continuous dynamics (1) are discretized as:

$$x_{k+1} = J_{A,\mu_k}\left(x_k - \mu_k B(x_k)\right)$$

with $\sum_{k=1}^{\infty} \mu_k = \infty$ and $\mu_k \to 0$ ensuring convergence analogous to the continuous case [2]. This is suitable for numerical optimization and control algorithms [1].

6.2 Engineering and Physical Applications

Monotone inclusions model mechanical equilibrium, control systems, electrical circuits, and dissipative physical systems. Our generalized framework supports robust numerical methods for these applications [4].

The generalized framework proposed in this paper enhances the applicability of monotone inclusions by removing restrictive assumptions, allowing for robust numerical solutions in various domains:

1. **Mechanical Equilibrium**: Monotone inclusions naturally arise in modeling mechanical systems under equilibrium, such as contact problems in solid mechanics. For instance, in frictional contact problems, the inclusion $0 \in A(x) + B(x)$ can represent the balance between contact forces (modeled by the maximal monotone operator A) and external forces or friction (modeled by B). The proposed method can be used to compute equilibrium configurations in structures like bridges or robotic arms, where nonlinear forces are prevalent.
2. **Control Systems**: In control theory, monotone inclusions are used to design robust controllers for nonlinear systems. For example, the framework can be applied to stabilize a robotic manipulator by solving for the control input that minimizes a cost function subject to constraints modeled as monotone inclusions.

The discrete version of the dynamics is particularly useful for real-time control applications, where iterative updates are computed at discrete time steps.

3. **Electrical Circuits**: Monotone inclusions model the behavior of nonlinear electrical circuits, such as those containing diodes or other nonlinear components. The operator A can represent the constitutive relations of passive components (e.g., resistors), while B models active or nonlinear elements (e.g., voltage sources or diodes). The proposed method enables the simulation of steady- state solutions in such circuits, ensuring stability and convergence even for complex networks [13].
4. **Dissipative Physical Systems**: Many physical systems, such as those governed by partial differential equations (PDEs) in fluid dynamics or heat transfer, exhibit dissipative behavior that can be modeled as monotone inclusions. The framework can be used to solve for steady-state solutions in systems with nonlinear dissipation, such as viscous flows or thermal conduction in composite materials.

6.3 Practical Example: Optimization in Machine Learning

A concrete application of the proposed framework is in training machine learning models, particularly in solving nonsmooth optimization problems. Consider a regularized empirical risk minimization problem, such as training a support vector machine (SVM) with a hinge loss function. The optimization problem can be formulated as:

$$\min_{x\in\mathbb{R}^n} f(x) + g(x)$$

where $f(x)$ is a smooth loss function (e.g., the hinge loss) and $g(x)$ is a nonsmooth regularizer (e.g., the l_1-norm for sparsity). This can be rewritten as a monotone inclusion $0 \in \partial g(x) + \nabla f(x)$), where ∂g is the subdifferential of g (a maximal monotone operator) and ∇f is the gradient of f (monotone and Lipschitz continuous). The proposed Newton-like dynamics can be applied to solve this inclusion iteratively, using the resolvent $J_{\partial g,\mu}$to handle the nonsmooth term. The discrete version of the algorithm can be implemented as:

$$x_{k+1} = J_{\partial g,\mu}\left(x_k - \mu_k \nabla f(x_k)\right)$$

where μ_k is a step size satisfying the convergence conditions. This approach is computationally efficient for large-scale datasets and can be extended to other machine learning tasks, such as deep learning with nonsmooth activation functions or sparse regression models [14].

6.4 Extensions to Adaptive Methods

The generalized framework paves the way for adaptive regularization techniques, such as Levenberg- Marquardt regularization, which dynamically adjust the regularization parameter $\mu(t)$ based on the problem's characteristics. These methods enhance convergence rates in applications where standard assumptions (e.g., strong monotonicity) do not hold, making the framework versatile for real-world problems with complex nonlinearities.

7. Conclusion

This paper significantly advances the study of monotone inclusions in Hilbert spaces by generalizing the Newton-like dynamics framework introduced in [5]. By removing structural constraints on the operators A and B, we have developed a more flexible and widely applicable method for solving monotone inclusions of the form $0 \in A(x) + B(x)$. The rigorous proofs of existence and uniqueness of strong solutions, combined with the weak and strong convergence results under well-defined

conditions, provide a robust theoretical foundation for the proposed dynamics. The use of Lyapunov functions and the resolvent operator $J_{A,\mu}$ enables a comprehensive analysis of convergence, addressing both theoretical and practical challenges.

The expanded applications section demonstrates the versatility of the framework in addressing real- world problems in mechanical equilibrium, control systems, electrical circuits, dissipative physical systems, and machine learning. By providing a concrete example in machine learning, we illustrate how the method can be applied to large-scale optimization problems, offering computational efficiency and robustness. The discrete version of the dynamics further enhances its practical utility, enabling implementation in numerical algorithms for optimization and control.

This work opens several avenues for future research. First, the framework can be extended to incorporate inertial or accelerated methods to improve convergence rates in cases where strong monotonicity does not hold. Second, adaptive regularization strategies, such as those inspired by Levenberg-Marquardt methods, can be further developed to handle a broader class of operators. Finally, the application of the framework to emerging fields, such as data science and networked systems, holds promise for addressing complex, high-dimensional problems. By bridging theoretical rigor with practical applicability, this work contributes to the advancement of applied mathematics and its impact on engineering, physics, and computational sciences.

References

[1] Attouch, H., Peypouquet, J.: Convergence of inertial dynamics and proximal algorithms governed by maximal monotone operators. J. Convex Anal., 27(2), 435–465 (2020).

[2] Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces, 2nd edn. Springer, Cham (2021).

[3] Peypouquet, J., Sezin, S.: Evolution equations with monotone operators: existence and convergence. SIAM J. Optim., 32(3), 1892–1917 (2022).

[4] Attouch, H., Cabot, A.: Asymptotic behavior of nonautonomous monotone and maximal monotone dynamics. J. Differential Equations, 345, 234–270 (2023).

[5] Abbas, B., Attouch, H., Svaizer, B.F.: Newton−like dynamics and forward−backward methods for structured monotone inclusions in Hilbert spaces. J. Optim. Theory Appl., 161, 331–360 (2014).

[6] Brezis, H.: Opérateurs Maximax Monotones et Semi−Groupes de Contractions dans les Espaces de Hilbert. North−Holland, Amsterdam (1973).

[7] Levenberg, K.: A method for the solution of certain non−linear problems in the least squares. Q. Appl. Math., 2, 164–168 (1944).

[8] Marquardt, D.W.: An algorithm for least squares estimation of non−linear parameters. SIAM J. Appl. Math., 11, 431–441 (1963).

[9] Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis. Springer, Berlin (2009).

[10] Bauschke, H.H., Borwein, J.M.: On projection algorithms for solving convex feasibility problems.
SIAM Rev., 38(3), 367–426 (1996).

[11] Combettes, P.L., Pesquet, J.-C.: Proximal splitting methods in signal processing. In: Fixed-Point Algorithms for Inverse Problems in Science and Engineering, pp. 185–212. Springer, New York (2011).

[12] Boyd, S., Vandenberghe, L.: Convex Optimization. Cambridge University Press, Cambridge (2004).

[13] Alaa Zakaria, Sellection of optimal features for image-based pattern recognition applications, Al. Baath University journal, volume , 2021.

[14] Alaa Zakaria, Yasser Khadra, Led Alabboud. Optimal of patterns in satellite images using Texture features, Al. Baath University journal, volume 40, 2018.