

الحل العددي لنظام من المعادلات التكاملية-التفاضلية الكسرية الخطية باستخدام طريقة المربعات الصغرى وكثيرات حدود Mamadu-Njoseh المزاحة

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□ ملخص □

نقدم في هذا العمل الحل العددي لمنظومة من المعادلات التكاملية-التفاضلية الكسرية الخطية وفق مفهوم مشتق كابوتو الكسري. تعتمد الطريقة العددية على استخدام طريقة المربعات الصغرى مع كثيرات حدود Mamadu-Njoseh المزاحة، حيث يتم تحويل المسألة إلى منظومة من المعادلات الجبرية الخطية. تم اختبار الطريقة العددية المقترحة من خلال حل ثلاث مسائل، حيث تشير المقارنات العددية مع الطرائق الأخرى إلى كفاءة ودقة الطريقة المقترحة. تم استخدام برنامج Mathematica للحصول على النتائج العددية والرسوم البيانية المرتبطة بالحل العددي والأخطاء الناتجة.

الكلمات المفتاحية: نظام من المعادلات التكاملية-التفاضلية الكسرية الخطية، مشتق كابوتو الكسري، طريقة المربعات الصغرى، كثيرات حدود Mamadu-Njoseh المزاحة.

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Numerical Solution of System of linear Fractional Integro-Differential Equations by Least Squares Method and the Shifted Mamadu-Njoseh Polynomials

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□ Abstract □

In this paper, we present a numerical solution for a system of linear fractional integro-differential equations (FIDEs). The fractional derivative is considered in the Caputo sense. The numerical method is based on the use of the least squares method with the aid of shifted Mamadu-Njoseh polynomials, where the problem is converted into a system of linear algebraic equations. The proposed numerical method was tested by solving three problems, where numerical comparisons with other methods indicate the efficiency and accuracy of the proposed method. Mathematica programming was used to process the numerical results, the graphs corresponding to the numerical solution, and the resulting errors.

Keywords: System of linear fractional integro-differential equations, Caputo fractional derivative, Least Squares Method, shifted Mamadu-Njoseh polynomials.

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1. Introduction

Fractional calculus (FC) is an extension of classical calculus with a history spanning over 300 years. It generalizes traditional calculus by introducing integration and differentiation of non-integer (fractional) order [1]. Fractional calculus has been used to model many natural phenomena across various fields, including natural sciences and engineering. Notable applications include earthquake modeling [2], control theory [3], electrical networks, fluid flow, diffusion, electromagnetic theory, probability, statistics, potential theory, and optics [4]. Fractional integro-differential equations have garnered substantial attention from mathematicians and physicists due to their effectiveness in modeling complex dynamical systems arising in engineering and scientific disciplines such as physics, chemistry, economics, electrochemistry, electromagnetism [5], convection and radiation problems, fluid dynamics, combined conduction, chemical kinetics, and biological models [6].

In recent years, numerous papers have focused on the development of numerical and analytical techniques for fractional integro-differential equations. For instance, Saeed and Sdeq [7] have used homotopy perturbation method to solve system of linear Fredholm fractional integro-differential equations. Saleh et.al [8] have used Adomian decomposition method to solve system of linear Fredholm fractional integro-differential equations. Deif and Grace [9] have developed a new technique based on iterative refinement to approximate the analytical solution of system of linear Fredholm fractional integro-differential equations. Zedan et.al [10] have applied Chebyshev spectral method to solve a system of fractional integro-differential equations and Abel's integral equations. Barazandeh [11] has used Müntz-Legendre wavelets and their operational matrix to solve a system of fractional integro-differential equations. Bargamadi et.al [12] have employed the second Chebyshev wavelets method and its operational matrix to solve system of fractional integro-differential equations with weakly singular kernels. Masoud et.al [13] have used tau method based on monic Laguerre polynomials to solve systems of fractional integro-differential equations. Rahrovi et.al [14] have used Jacobi wavelets and its operational matrix to solve nonlinear fractional population growth model. Henka et.al [15] have applied Hermite wavelets collocation method to solve fractional Fredholm integro-differential equations.

In this paper, we present a numerical solution of a system of linear fractional integro-differential equation of the form:

$$D^\alpha u_r(x) + \sum_{k=1}^l p_{rk}(x) u_k(x) = f_r(x) + \int_0^1 K_r(x, t) \left(\sum_{k=1}^l \mu_{rk} u_k(t) \right) dt \quad (1)$$

$$; r = 1, 2, \dots, l, 0 \leq x, t \leq 1, \alpha > 0,$$

with the initial conditions

$$u_r^i(0) = u_{ri}, r = 1, 2, \dots, l, i = 0, 1, \dots, [\alpha] - 1, \quad (2)$$

where $D^\alpha u_r(x)$ denotes the α -th fractional derivative of $u_r(x)$, $p_{rk}(x)$ and $f_r(x)$ are known continuous functions, $K_r: [0,1] \times [0,1] \rightarrow \mathbb{R}$, $r = 1, 2, \dots, l$ are known continuous functions. Here, x and t are real variables in interval $[0,1]$, μ_{rk} are real constants, and $u_r(x)$ is the unknown functions to be determined.

2. Importance and Objectives of the Research:

This study focuses on obtaining a numerical solution for a system of linear fractional integro-differential equations using the least squares method and shifted Mamadu-Njoseph Polynomials, Additionally, we assess the accuracy and efficiency of the proposed method. Such equations play a fundamental role in modeling a wide range of phenomena in science and engineering.

3. Methodology:

This research falls within the discipline of applied mathematics, specifically in the field of fractional integro-differential equations and numerical analysis. The mathematical techniques employed are based on approximation theory, linear algebraic systems, and computational software such as Mathematica 12.

4. Basic definitions of fractional calculus:

In this section, we present some basic definitions and properties of fractional calculus which are used further in this paper.

Definition 4.1 (see [16]). Riemann–Liouville fractional integral

The Riemann–Liouville fractional integral operator of order α for a function u is given by:

$$J^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s) ds, \quad \alpha >$$

$$0 \quad (3)$$

where $J^0 u(x) = u(x)$, $m-1 < \alpha \leq m$, $m \in \mathbb{N}^*$, and $\Gamma(\cdot)$ is the Gamma function.

Definition 4.2 (see [17]). Caputo's fractional derivative

The Caputo fractional derivative operator of order α for a function u is given by:

$$D^\alpha u(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} u^{(m)}(s) ds \quad (4)$$

where $m-1 < \alpha \leq m$, $m \in \mathbb{N}^*$, $x > 0$.

Hence, we have the following properties [16]:

- $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}$, $\beta > -1$.
- $J^\alpha D^\alpha u(x) = u(x) - \sum_{k=0}^{|\alpha|-1} u^{(k)}(0^+) \frac{x^k}{k!}$; $x > 0$.
- $D^\alpha J^\alpha u(x) = u(x)$.
- $D^\alpha C = 0$, C is constant.

$$\bullet \quad D^\alpha x^\beta = \begin{cases} 0 & ; \beta \in \mathbb{N}_0, \beta < [\alpha]. \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha} & ; \beta \in \mathbb{N}_0, \beta \geq [\alpha]. \end{cases}$$

5. Basic properties of modified Mamadu-Njoseh Polynomials

The Mamadu-Njoseh polynomials of degree n are defined on the closed interval $[-1,1]$ by: [18]

$$\varphi_n(x) = \sum_{i=0}^n C_i^{(n)} x^i \tag{5}$$

where $\varphi_j(x), j = 0,1,2,3 \dots$, are orthogonal polynomials on the interval $[-1,1]$ with respect to the weight function $w(x) = x^2 + 1$.

Mamadu-Njoseh polynomials are shifted on the closed interval $[0,1]$ as follows:

$\varphi_j^*(x) = \varphi_j(2x - 1), j = 0,1,2,3 \dots$, where $\varphi_j^*(x), (j = 0,1,2,3 \dots)$ are called the shifted Mamadu-Njoseh polynomials. The polynomials from degree one up to seven are given as follows:

$$\left. \begin{aligned} \varphi_0^*(x) &= 1, \\ \varphi_1^*(x) &= 2x - 1, \\ \varphi_2^*(x) &= \frac{1}{3}(1 - 20x + 20x^2), \\ \varphi_3^*(x) &= \frac{1}{5}(-5 + 66x - 168x^2 + 112x^3), \\ \varphi_4^*(x) &= \frac{1}{9}(9 - 196x + 910x^2 - 1428x^3 + 238x^4), \\ \varphi_5^*(x) &= \frac{1}{17}(-17 + 550x - 3990x^2 + 10800x^3 - 12210x^4 + 4884x^5), \\ \varphi_6^*(x) &= \frac{1}{133}(333 - 5976x + 62010x^2 - 252780x^3 + 478170x^4 - 422136x^5 + 140712x^6) \end{aligned} \right\} \tag{6}$$

we creat The functions $\bar{u}_{n,r}(x), (r = 1,2, \dots, l)$ by shifted Mamadu-Njoseh polynomials, as follows:

$$\bar{u}_{n,r}(x) = \sum_{i=0}^n a_i^r \varphi_i^*(x) = A_r^T \Psi(x), r = 1,2, \dots, l, \tag{7}$$

where

$$A_r^T = [a_0^r, a_1^r, \dots, a_n^r] \quad ; \quad a_i^r \in \mathbb{R} \quad ; \quad i = 0,1, \dots, n, r = 1,2, \dots, l.$$

$$\Psi(x) = [\varphi_0^*(x), \varphi_1^*(x), \dots, \varphi_n^*(x)]^T.$$

6. Mathematical Formulation of Proposed Method

In this section, the least squares method (LSM) with aid of shifted Mamadu-Njoseh polynomials are applied to find the numerical solution of system (1)-(2). Consider Eq. (1) operating with J^α on both sides as follows:

$$J^\alpha D^\alpha u_r(x) + J^\alpha \left[\sum_{k=1}^l p_{rk}(x) u_k(x) \right] = J^\alpha f_r(x) + J^\alpha \left[\int_0^1 K_r(x, t) \left(\sum_{k=1}^l \mu_{rk} u_k(t) \right) dt \right] \tag{8}$$

; $r = 1,2, \dots, l$.

We get

$$u_r(x) - \sum_{S=0}^{m-1} u_r^S(0) \frac{x^S}{S!} + J^\alpha \left[\sum_{k=1}^l p_{rk}(x) u_k(x) \right] \\ = J^\alpha f_r(x) + J^\alpha \left[\int_0^1 K_r(x,t) \left(\sum_{k=1}^l \mu_{rk} u_k(t) \right) dt \right], \quad r = 1, 2, \dots, l. \quad (9)$$

The function $\bar{u}_r(x)$ can be created as a linear construct of shifted Mamadu-Njoseh polynomials, as follows:

$$\bar{u}_{n,r}(x) = \sum_{i=0}^n a_i^r \varphi_i^*(x), \quad r = 1, 2, \dots, l. \quad (10)$$

Let $\bar{u}_{n,r}(x)$ denote approximate solution of exact solution $u_r(x)$, Substituting (10) into (9), we obtain:

$$\sum_{i=0}^n a_i^r \varphi_i^*(x) - \sum_{S=0}^{m-1} u_r^S(0) \frac{x^S}{S!} + J^\alpha \left[\sum_{k=1}^l p_{rk}(x) \left(\sum_{i=0}^n a_i^k \varphi_i^*(x) \right) \right] \\ = J^\alpha f_r(x) + J^\alpha \left[\int_0^1 K_r(x,t) \left(\sum_{k=1}^l \mu_{rk} \left(\sum_{i=0}^n a_i^k \varphi_i^*(t) \right) \right) dt \right], \quad r = 1, 2, \dots, l. \quad (11)$$

Hence, $R_r(x, a_0^r, a_1^r, \dots, a_n^r)$ denotes the residual equation of system (1)-(2) is given by:

$$R_r(x, a_0^r, a_1^r, \dots, a_n^r) = \sum_{i=0}^n a_i^r \varphi_i^*(x) - \sum_{S=0}^{m-1} u_r^S(0) \frac{x^S}{S!} \\ + J^\alpha \left[\sum_{k=1}^l p_{rk}(x) \left(\sum_{i=0}^n a_i^k \varphi_i^*(x) \right) \right] - J^\alpha f_r(x) \\ - J^\alpha \left[\int_0^1 K_r(x,t) \left(\sum_{k=1}^l \mu_{rk} \left(\sum_{i=0}^n a_i^k \varphi_i^*(t) \right) \right) dt \right], \quad r = 1, 2, \dots, l. \quad (12)$$

Letting:

$$S_r(a_0^r, a_1^r, \dots, a_n^r) = \int_0^1 [R_r(x, a_0^r, a_1^r, \dots, a_n^r)]^2 w(x) dx, \quad r = 1, 2, \dots, l \quad (13)$$

where $w(x)$ is the positive weight function defined on the interval $[0,1]$. Here, we take $w(x) = 1$ for simplicity. Thus:

$$\begin{aligned}
 S_r(a_0^r, a_1^r, \dots, a_n^r) &= \\
 &= \int_0^1 \left(\sum_{i=0}^n a_i^r \varphi_i^*(x) \right. \\
 &\quad \left. - \sum_{s=0}^{m-1} u^s(0) \frac{x^s}{s!} + J^\alpha \left[\sum_{k=1}^l p_{rk}(x) \left(\sum_{i=0}^n a_i^k \varphi_i^*(x) \right) \right] - J^\alpha f_r(x) \right. \\
 &\quad \left. - J^\alpha \left[\int_0^1 K_r(x, t) \left(\sum_{k=1}^l \mu_{rk} \left(\sum_{i=0}^n a_i^k \varphi_i^*(t) \right) \right) dt \right] \right)^2 dx. \tag{14}
 \end{aligned}$$

In order to minimize the functions (14), we obtained the coefficients of $a_i^r (i = 0, 1, \dots, n)$ as follows:

$$\frac{\partial S_r}{\partial a_j^r} = 0 \quad ; \quad j = 0, 1, \dots, n, \quad r = 1, 2, \dots, l \tag{15}$$

then, we have:

$$\begin{aligned}
 &\int_0^1 \left(\sum_{i=0}^n a_i^r \varphi_i^*(x) - \sum_{s=0}^{m-1} u^s(0) \frac{x^s}{s!} + J^\alpha \left[\sum_{k=1}^l p_{rk}(x) \left(\sum_{i=0}^n a_i^k \varphi_i^*(x) \right) \right] - J^\alpha f_r(x) \right. \\
 &\quad \left. - J^\alpha \left[\int_0^1 K_r(x, t) \left(\sum_{k=1}^l \mu_{rk} \left(\sum_{i=0}^n a_i^k \varphi_i^*(t) \right) \right) dt \right] \right) dx \\
 &\quad \times \int_0^1 \left(\varphi_j^*(x) + J^\alpha [p_{rr}(x) \varphi_j^*(x)] - J^\alpha \left[\int_0^1 K_r(x, t) \mu_{rr} \varphi_j^*(t) dt \right] \right) dx = 0 \tag{16}
 \end{aligned}$$

for all $r = 1, 2, \dots, l$, we can obtain a system of $(n + 1)$ linear equations with $l(n + 1)$ unknown coefficients $a_i^k (i = 0, 1, \dots, n, k = 1, 2, \dots, l)$. This system can be formed by using matrices form as follows:

$$H^r \beta^r = G^r, \quad r = 1, 2, \dots, l \tag{17}$$

where H^r is square matrix of order $(n + 1)$, β^r and G^r are column vectors of order $(n + 1)$. For all $0 \leq i, j \leq n$, we have

$$\begin{cases}
 H^r_{ji} = \int_0^1 \aleph_i^r(x) \xi_j^r(x) dx, \\
 \beta^r = (1, 1, \dots, 1)^T, \\
 G^r_j = \int_0^1 \left[J^\alpha f_r(x) + \sum_{s=0}^{m-1} u^s(0) \frac{x^s}{s!} \right] \xi_j^r(x) dx,
 \end{cases} \tag{18}$$

where

$$\aleph_i^r(x) = a_i^r \varphi_i^*(x) + J^\alpha \left[\sum_{k=1}^l p_{rk}(x) a_i^k \varphi_i^*(x) \right] - J^\alpha \left[\int_0^1 K_r(x, t) \left(\sum_{k=1}^l \mu_{rk} a_i^k \varphi_i^*(t) \right) dt \right].$$

$$\xi_j^r(x) = \varphi_j^*(x) + J^\alpha [p_{rr}(x) \varphi_j^*(x)] - J^\alpha \left[\int_0^1 K_r(x, t) \mu_{rr} \varphi_j^*(t) dt \right].$$

By solving the above linear system (17) we obtain the values of the unknown coefficients $a_i^r (i = 0, 1, \dots, n, r = 1, 2, \dots, l)$. Thus, we get the approximate solution of problem (1)-(2).

7. Error Estimate

Theorem 7.1. Let $u(x) = [u_1(x), u_2(x), \dots, u_l(x)]^T \in R^l$ and $\bar{u}(x) = [\bar{u}_1(x), \bar{u}_2(x), \dots, \bar{u}_l(x)]^T \in R^l$ be the exact and approximate solution vectors of the system (1)-(2), respectively. For each $r = 1, 2, \dots, l$, suppose that $u_r(x) \in C^{n+1}([0, 1])$ and $\bar{u}_{n,r}(x) = A_r^T \Psi(x)$ are linear combination of shifted Mamadu-Njoseh polynomials. Then, for $x \in [0, 1]$,

$$\|u(x) - \bar{u}_n(x)\|_2 \leq \frac{\delta}{2^{2n+1} (n+1)!}, \quad (19)$$

where $\delta = \sum_{r=1}^l \delta_r$; δ_r is a constant such that $\delta_r = \text{Max}_{x \in [0, 1]} |u_r^{(n+1)}(x)|$, $r = 1, 2, \dots, l$.

Proof. For each $r = 1, 2, \dots, l$, let $p_{n,r}(x)$ denote the interpolating polynomial of $u_r(x)$ at the nodes $\{x_s, u_r(x_s)\}_{s=0}^n$, where $x_s (s = 0, 1, \dots, n)$ are the roots of the shifted Chebyshev polynomial of degree $n+1$ on the interval $[0, 1]$. Then, according to the interpolation error, we obtain:

$$u_r(x) - p_{n,r}(x) = \frac{u_r^{(n+1)}(\gamma)}{(n+1)!} \prod_{s=0}^n (x - x_s); \quad \gamma \in [0, 1], \quad r = 1, 2, \dots, l.$$

The estimates for Chebyshev interpolation nodes are that [19]:

$$|u_r(x) - p_{n,r}(x)| \leq \frac{\delta_r}{2^{2n+1} (n+1)!}; \quad \forall x \in [0, 1], \quad r = 1, 2, \dots, l. \quad (20)$$

Since $\bar{u}_{n,r}(x)$ is the unique best approximation of $u_r(x)$, then we find that:

$$\begin{aligned} \|u_r(x) - \bar{u}_{n,r}(x)\|_2^2 &\leq \|u_r(x) - p_{n,r}(x)\|_2^2 \\ &= \int_0^1 |u_r(x) - p_{n,r}(x)|^2 dx = \int_0^1 \left(\frac{\delta_r}{2^{2n+1} (n+1)!} \right)^2 dx = \left(\frac{\delta_r}{2^{2n+1} (n+1)!} \right)^2, \end{aligned}$$

then,

$$\|u_r(x) - \bar{u}_{n,r}(x)\|_2 \leq \frac{\delta_r}{2^{2n+1} (n+1)!}, \quad r = 1, 2, \dots, l.$$

Therefore, the total truncation error is:

$$\|u(x) - \bar{u}_n(x)\|_2 = \sum_{r=1}^l \|u_r(x) - \bar{u}_{n,r}(x)\|_2 \leq \sum_{r=1}^l \frac{\delta_r}{2^{2n+1} (n+1)!} \leq \frac{\delta}{2^{2n+1} (n+1)!}; \quad \delta = \sum_{r=1}^l \delta_r.$$

Thus, the proof of this theorem is completed.

8. Numerical Examples

In this section, we test the effectiveness of the proposed method by solving three examples. The numerical results and graphs corresponding to the numerical solution and the resulting errors were processed by using the Wolfram Mathematica software.

we denote absolute error as:

$$absolute\ error = |u_r(x) - \bar{u}_{n,r}(x)| ; 0 \leq x \leq 1, r = 1, 2, \dots, l,$$

where $u_r(x)$ is the exact solution and $\bar{u}_{n,r}(x)$ is the approximate solution.

Example 8.1. [8]: Consider the following system of fractional integro-differential equation:

$$\begin{cases} D^{\frac{2}{3}} u_1(x) = \frac{3}{2} \frac{x^{\frac{1}{3}} \sqrt{3} \Gamma(\frac{2}{3})}{\pi} - \frac{1}{6} x + \int_0^1 2 x t (u_1(x) + u_2(x)) dt, \\ D^{\frac{2}{3}} u_2(x) = \frac{9}{4} \frac{x^{\frac{4}{3}} \sqrt{3} \Gamma(\frac{2}{3})}{\pi} + \frac{5}{6} x^3 + \int_0^1 x^3 (u_1(x) - u_2(x)) dt, \end{cases} \quad (21)$$

, $0 \leq x, t \leq 1$, with the initial conditions $u_1(0) = -1, u_2(0) = 0$, the exact solutions are

$$u_1(x) = x - 1, \quad u_2(x) = x^2.$$

Solution: Applying the proposed method for $(n = 3)$, we get the values of the coefficients as:

$$\begin{cases} a_0^1 = -0.50000000000000033, a_1^1 = 0.49999999999999933, \\ a_2^1 = -4.267438841479167 \times 10^{-15}, a_3^1 = -4.957308590590608 \times 10^{-15}. \end{cases} \quad (22)$$

$$\begin{cases} a_0^2 = 0.449999999999999896, a_1^2 = 0.50000000000000001, \\ a_2^2 = 0.14999999999999987, a_3^2 = 6.348710869160229 \times 10^{-16}. \end{cases} \quad (23)$$

These values are then substituted into equation (10), after simplifying we obtain the approximate solution as:

$$\begin{aligned} \bar{u}_{3,1}(x) = & -0.9999999999999991 + 0.99999999999999617 x + 1.3811597636731667 \times 10^{-13} x^2 \\ & - 1.1104371242922963 \times 10^{-13} x^3. \end{aligned} \quad (24)$$

$$\begin{aligned} \bar{u}_{3,2}(x) = & -2.22738494315422 \times 10^{-15} + 1.7430501486614958 \times 10^{-14} x \\ & + 0.99999999999999698 x^2 + 1.422111234691891 \times 10^{-14} x^3. \end{aligned} \quad (25)$$

Table 1. Comparisons of the approximate solution with the exact solution and also displays the absolute errors for Example 8.1.

			(n = 3) Present Method			
<i>x</i>	<i>Exact Solution</i> $u_1(x)$	<i>Exact Solution</i> $u_2(x)$	<i>Approximate Solution</i> $\bar{u}_{3,1}(x)$	<i>Absolute Error</i> $\bar{u}_{3,1}(x)$	<i>Approximate Solution</i> $\bar{u}_{3,2}(x)$	<i>Absolute Error</i> $\bar{u}_{3,2}(x)$
0.1	-0.9	0.01	-0.90000000000000017	1.77636 E-15	0.00999999999999918	8.18789 E-16

0.2	-0.8	0.04	-0.80000000000000022	2.22045 E-15	0.04000000000000002	1.94289 E-16
0.3	-0.7	0.09	-0.70000000000000012	1.11022 E-15	0.090000000000000064	6.38378 E-16
0.4	-0.6	0.16	-0.59999999999999994	5.55112 E-16	0.160000000000000092	8.88178 E-16
0.5	-0.5	0.25	-0.49999999999999976	2.44249 E-15	0.25000000000000007	7.21645 E-16
0.6	-0.4	0.36	-0.399999999999999636	3.66374 E-15	0.36000000000000003	3.33067 E-16
0.7	-0.3	0.49	-0.299999999999999644	3.55271 E-15	0.49000000000000005	1.11022 E-16
0.8	-0.2	0.64	-0.199999999999999818	1.77636 E-15	0.63999999999999999	2.22045 E-16
0.9	-0.1	0.81	-0.10000000000000027	2.66454 E-15	0.80999999999999996	4.44089 E-16
1.0	0	1.0	-0.000000000000001038	1.04361 E-14	0.99999999999999992	7.77156 E-16

Table .2. describes the absolute errors obtained in [8].

<i>Absolute Errors of Adomian decomposition method [8]</i>		
<i>(n = 3)</i>		
<i>x</i>	<i>Absolute Error</i> $\bar{u}_{3,1}(x)$	<i>Absolute Error</i> $\bar{u}_{3,2}(x)$
0.1	7.77629 E-5	7.8584 E-8
0.2	2.468819 E-4	9.9796 E-7
0.3	4.852601 E-4	4.41348 E-6
0.4	7.838011 E-4	1.26733 E-5
0.5	1.1369007 E-3	2.87228 E-5
0.6	1.5406049 E-3	5.60477 E-5
0.7	1.9919080 E-3	9.86345 E-5
0.8	2.4884133 E-3	1.609408 E-4
0.9	3.0281453 E-3	2.478708 E-4
1.0	3.609435 E-3	3.647564 E-4

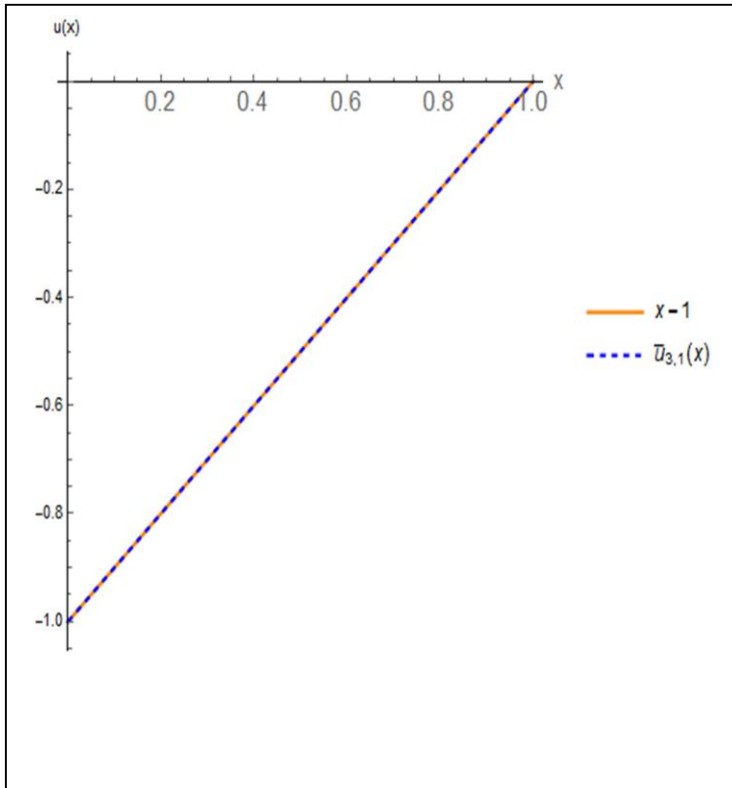


Figure 1: A comparisons between the exact and approximate solution of $u_1(x)$ for $(n = 3)$. (Example 8.1)

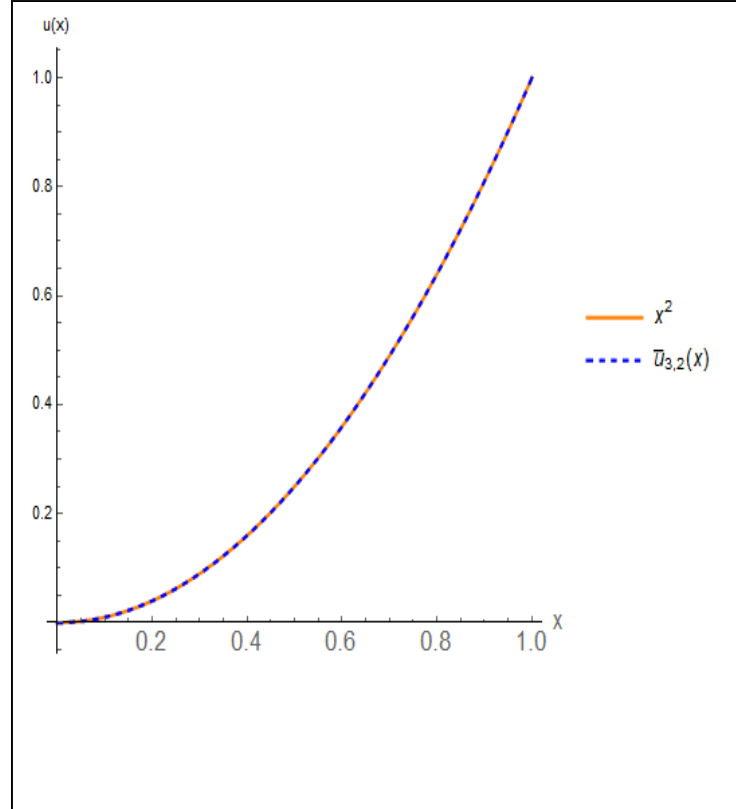


Figure 2: A comparisons between the exact and approximate solution of $u_2(x)$ for $(n = 3)$. (Example 8.1)

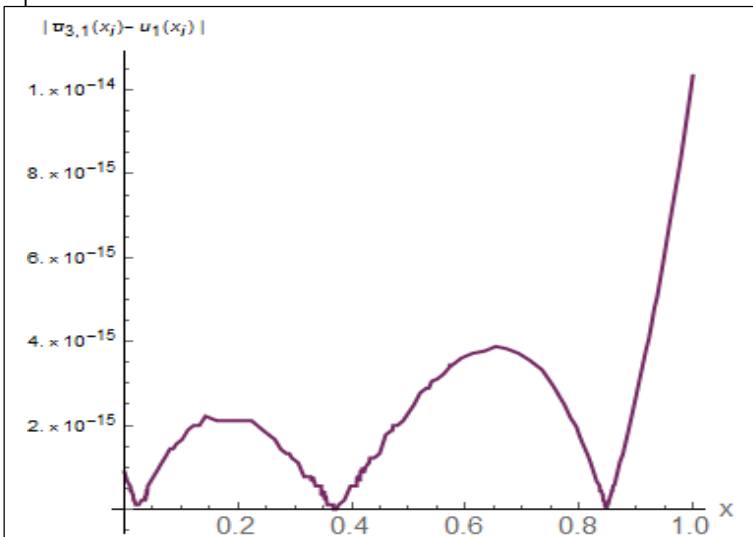


Figure 3: The absolute errors of the numerical solution of $\bar{u}_{3,1}(x)$ for $(n = 3)$. (Example 8.1)

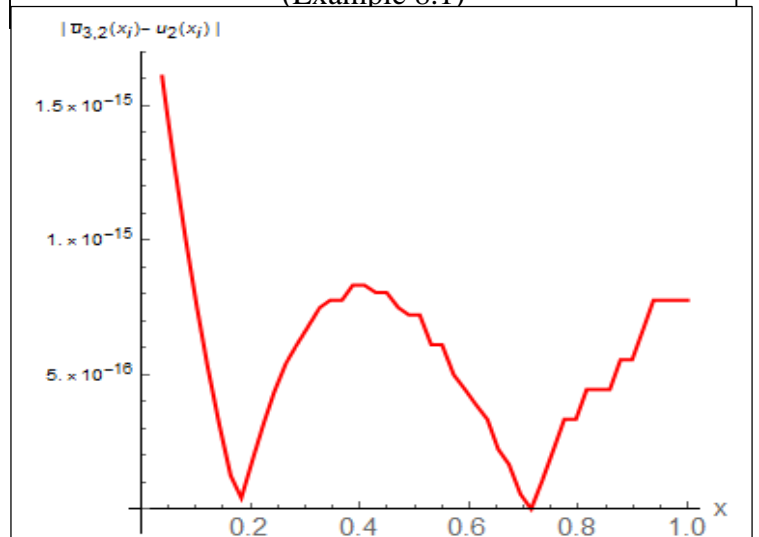


Figure 4: The absolute errors of the numerical solution of $\bar{u}_{3,2}(x)$ for $(n = 3)$. (Example 8.1)

Example 8.2. [7]: Consider the following system of fractional integro-differential equation:

$$\begin{cases} D^{\frac{1}{2}} u_1(x) = \frac{2\sqrt{x}}{\sqrt{\pi}} - \frac{5}{6} + \int_0^1 (u_1(t) + u_2(t)) dt, \\ D^{\frac{3}{2}} u_2(x) = \frac{4\sqrt{x}}{\sqrt{\pi}} - \frac{x}{6} + \int_0^1 x(u_1(t) - u_2(t)) dt, \end{cases} \quad (26)$$

, $0 \leq x, t \leq 1$, with the initial conditions $u_1(0) = 0, u_2(0) = 0$, the exact solutions are $u_1(x) = x, u_2(x) = x^2$.

Solution: Applying the proposed method for $(n = 3)$, we get the values of the coefficients as:

$$\begin{cases} a_0^1 = 0.50000000000000019, a_1^1 = 0.50000000000000002, \\ a_2^1 = 3.446428177401212 \times 10^{-17}, a_3^1 = -1.081683337748663 \times 10^{-18}. \end{cases} \quad (27)$$

$$\begin{cases} a_0^2 = 0.449999999999999685, a_1^2 = 0.50000000000000002, \\ a_2^2 = 0.149999999999999583, a_3^2 = 2.092690934642706 \times 10^{-15}. \end{cases} \quad (28)$$

These values are then substituted into equation (10), after simplifying we obtain the approximate solution as:

$$\bar{u}_{3,1}(x) = 1.677904314200154 \times 10^{-15} + 1.00000000000000002 x + 2.661064386417692 \times 10^{-16} x^2 - 2.4229706765570074 \times 10^{-17} x^3. \quad (29)$$

$$\bar{u}_{3,2}(x) = -6.862566070964249 \times 10^{-15} + 5.595524044110789 \times 10^{-14} x + 0.99999999999999019 x^2 + 4.687627693599663 \times 10^{-14} x^3. \quad (30)$$

Table 3. Comparisons of the approximate solution with the exact solution and also displays the absolute errors for Example 8.2.

			$(n = 3)$ Present Method			
x	Exact Solution $u_1(x)$	Exact Solution $u_2(x)$	Approximate Solution $\bar{u}_{3,1}(x)$	Absolute Error $\bar{u}_{3,1}(x)$	Approximate Solution $\bar{u}_{3,2}(x)$	Absolute Error $\bar{u}_{3,2}(x)$
0.1	0.1	0.01	0.100000000000000168	1.67921 E-15	0.009999999999997788	2.21351 E-15
0.2	0.2	0.04	0.200000000000000176	1.7486 E-15	0.04000000000000083	8.18789 E-16
0.3	0.3	0.09	0.300000000000000177	1.77636 E-15	0.090000000000000233	2.33147 E-15
0.4	0.4	0.16	0.40000000000000018	1.77636 E-15	0.160000000000000283	2.80331 E-15
0.5	0.5	0.25	0.50000000000000019	1.88738 E-15	0.250000000000000244	2.44249 E-15

0.6	0.6	0.36	0.60000000000000019	1.88738 E-15	0.36000000000000014	1.38778 E-15
0.7	0.7	0.49	0.70000000000000002	1.9984 E-15	0.49000000000000002	2.77556 E-16
0.8	0.8	0.64	0.80000000000000002	1.9984 E-15	0.63999999999999992	8.88178 E-16
0.9	0.9	0.81	0.90000000000000021	2.10942 E-15	0.80999999999999983	1.77636 E-15
1.0	1.0	1.0	1.00000000000000022	2.22045 E-15	0.99999999999999978	2.22045 E-15

Table .4. describes the absolute errors obtained in [7].

Absolute errors of homotopy perturbation method [7]				
	(n = 2)		(n = 6)	
x	Absolute Error $\bar{u}_{2,1}(x)$	Absolute Error $\bar{u}_{2,2}(x)$	Absolute Error $\bar{u}_{6,1}(x)$	Absolute Error $\bar{u}_{6,2}(x)$
0.2	1.703 E-1	2.319 E-6	1.4219 E-7	1.0926 E-6
0.4	3.818 E-1	1.300 E-3	4.0217 E-7	6.1805 E-6
0.6	4.677 E-1	3.610 E-2	7.4485 E-7	1.700 E-3
0.8	5.400 E-1	7.420 E-2	1.1375 E-6	3.460 E-2
1.0	6.037 E-1	1.2920 E-1	1.5897 E-6	6.080 E-2

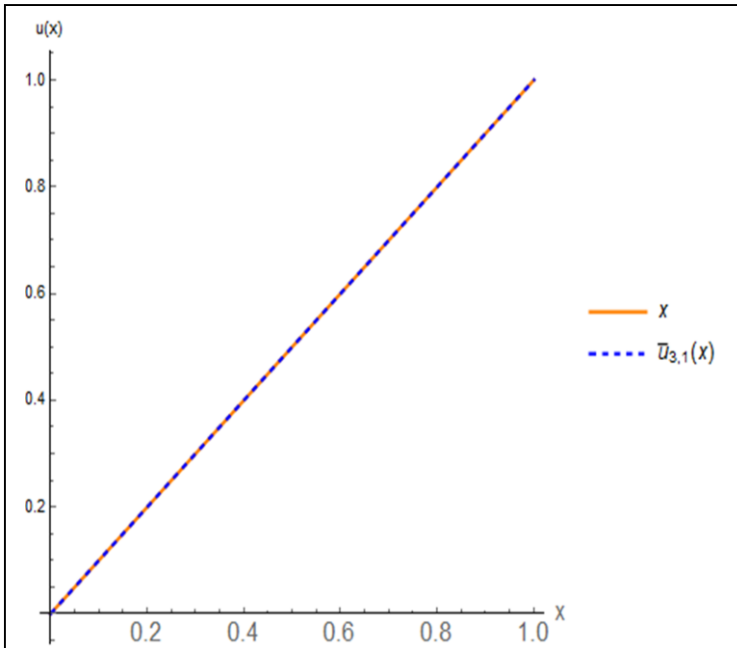


Figure 5: A comparisons between the exact and approximate solution of $u_1(x)$ for $(n = 3)$. (Example 8.2)

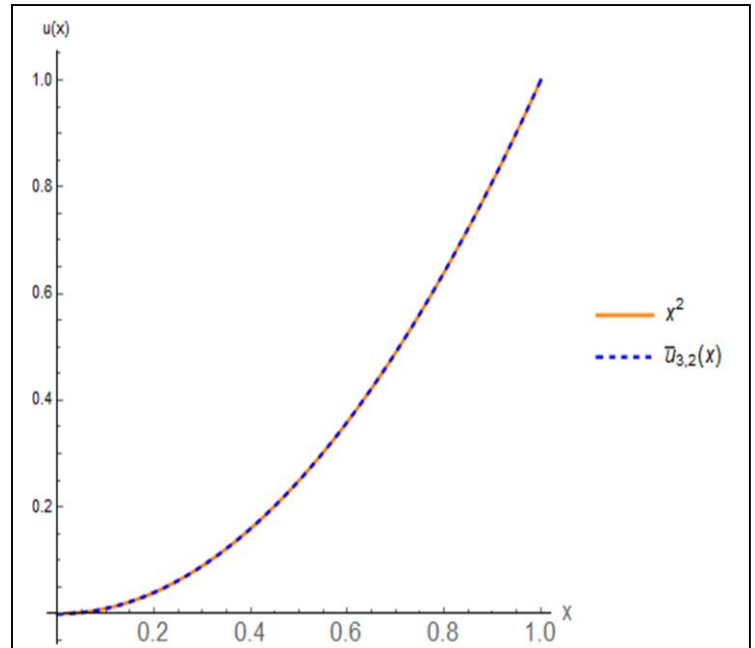


Figure 6: A comparisons between the exact and approximate solution of $u_2(x)$ for $(n = 3)$. (Example 8.2)

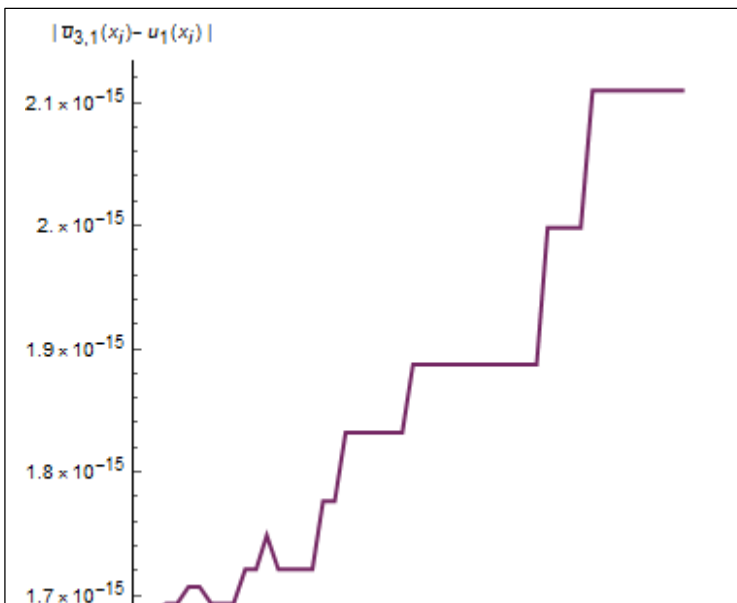


Figure 7: The absolute errors of the numerical solution of $\bar{u}_{3,1}(x)$ for $(n = 3)$. (Example 8.2)

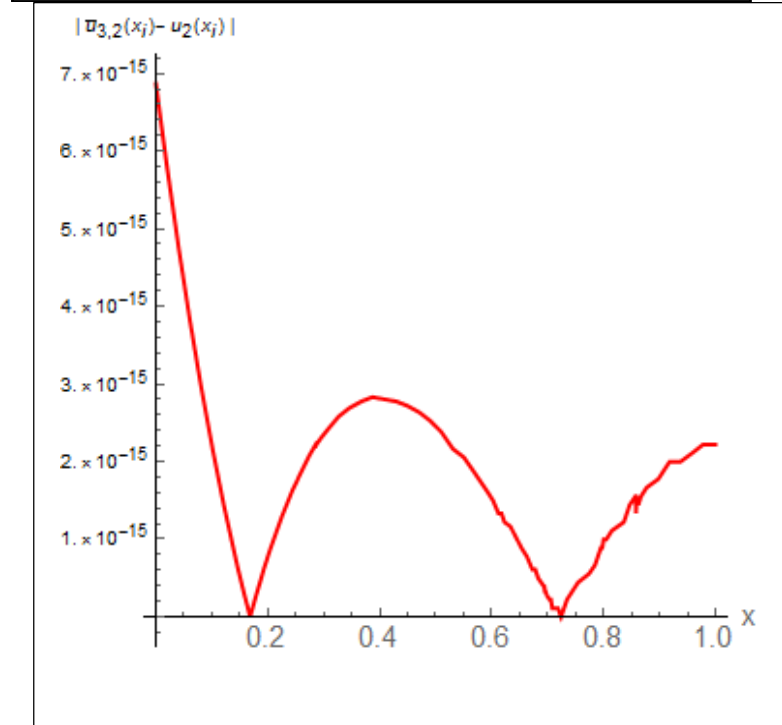


Figure 8: The absolute errors of the numerical solution of $\bar{u}_{3,2}(x)$ for $(n = 3)$. (Example 8.2)

$$\begin{cases} D^{0.5} u_1(x) = \frac{2\sqrt{x}}{\sqrt{\pi}} - \frac{x}{2} + \int_0^1 x u_2(x) dt, \\ D^{0.5} u_2(x) = \frac{2\sqrt{x}}{\sqrt{\pi}} - \frac{1}{3} + \int_0^1 t u_1(x) dt, \end{cases} \quad (31)$$

, $0 \leq x, t \leq 1$, with the initial conditions $u_1(0) = 0$, $u_2(0) = 0$, the exact solutions are $u_1(x) = x$, $u_2(x) = x$.

Solution: Applying the proposed method for $(n = 3)$, we get the values of the coefficients as:

$$\begin{cases} a_0^1 = 0.50000000000000007, a_1^1 = 0.5, \\ a_2^1 = 6.726501124136422 \times 10^{-16}, a_3^1 = -1.555809373523603 \times 10^{-17}. \end{cases} \quad (32)$$

$$\begin{cases} a_0^2 = 0.50000000000000002, a_1^2 = 0.49999999999999994, \\ a_2^2 = 3.002831050605498 \times 10^{-16}, a_3^2 = 8.26058798084202 \times 10^{-18}. \end{cases} \quad (33)$$

These values are then substituted into equation (10), after simplifying we obtain the approximate solution as:

$$\bar{u}_{3,1}(x) = 9.059086126482105 \times 10^{-16} + 0.9999999999999953 x + 5.007086032261545 \times 10^{-15} x^2 - 3.485012996692871 \times 10^{-16} x^3. \quad (34)$$

$$u_{3,2}(x) = 3.693895365289637 \times 10^{-16} + 0.9999999999999998 x + 1.72433161091404 \times 10^{-15} x^2 + 1.850371707708612 \times 10^{-16} x^3. \quad (35)$$

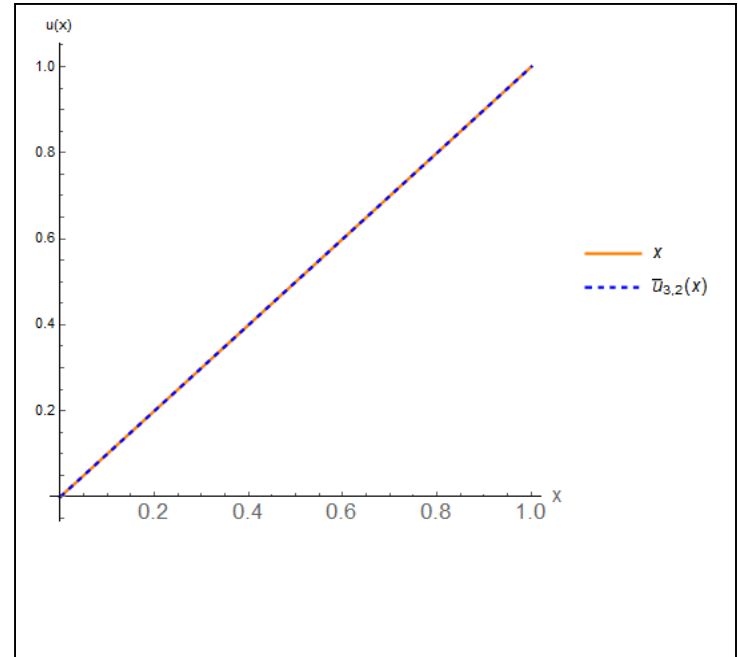
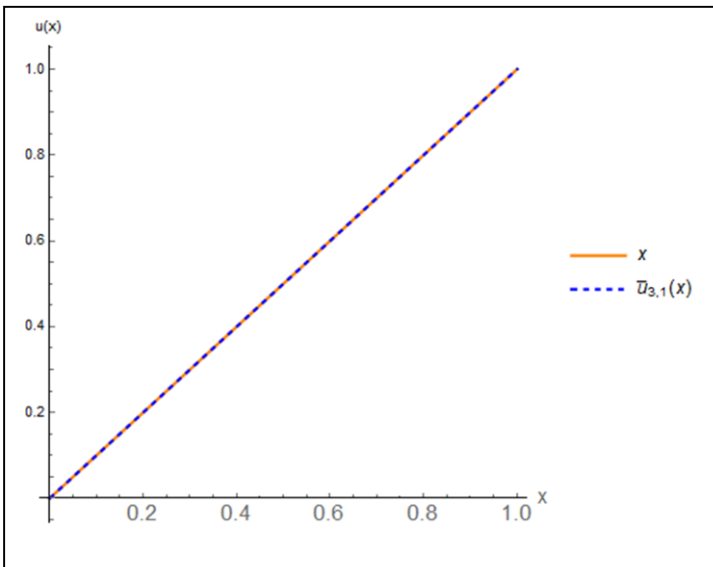


Figure 9: A comparisons between the exact and approximate solution of $u_1(x)$ for $(n = 3)$ (Example 8.3).

Table 5. Comparisons of the approximate solution with the exact solution and also displays the absolute errors for Example 8.3.

			<i>(n = 3) Present Method</i>			
<i>x</i>	<i>Exact solution</i> $u_1(x)$	<i>Exact solution</i> $u_2(x)$	<i>Approximate solution</i> $\bar{u}_{3,1}(x)$	<i>Absolute error</i> $\bar{u}_{3,1}(x)$	<i>Approximate solution</i> $\bar{u}_{3,2}(x)$	<i>Absolute error</i> $\bar{u}_{3,2}(x)$
0.1	0.1	0.1	0.100000000000000046	4.57967 E-16	0.100000000000000017	1.66533 E-16
0.2	0.2	0.2	0.200000000000000018	1.66533 E-16	0.200000000000000007	5.55112 E-17
0.3	0.3	0.3	0.29999999999999993	5.55112 E-17	0.29999999999999993	5.55112 E-17
0.4	0.4	0.4	0.39999999999999986	1.66533 E-16	0.39999999999999986	1.66533 E-16
0.5	0.5	0.5	0.49999999999999998	2.22045 E-16	0.49999999999999983	1.66533 E-16
0.6	0.6	0.6	0.59999999999999998	2.22045 E-16	0.59999999999999999	1.11022 E-16
0.7	0.7	0.7	0.7	0	0.69999999999999998	1.11022 E-16
0.8	0.8	0.8	0.80000000000000003	2.22045 E-16	0.8	0
0.9	0.9	0.9	0.90000000000000005	4.44089 E-16	0.9	0
1.0	1.0	1.0	1.00000000000000009	8.88178 E-16	1.00000000000000002	2.22045 E-16

Table .6. describes the absolute errors obtained in [7].

<i>Absolute errors of homotopy perturbation method [7]</i>				
	<i>(n = 2)</i>		<i>(n = 6)</i>	
<i>x</i>	<i>Absolute Error</i> $\bar{u}_{2,1}(x)$	<i>Absolute Error</i> $\bar{u}_{2,2}(x)$	<i>Absolute Error</i> $\bar{u}_{6,1}(x)$	<i>Absolute Error</i> $\bar{u}_{6,2}(x)$
0.2	5.4392 E-6	2.700 E-3	1.4219 E-7	7.1093 E-7
0.4	1.500 E-3	3.870 E-2	4.0217 E-7	1.0054 E-6
0.6	2.800 E-3	4.690 E-2	7.4485 E-7	1.2314 E-6
0.8	4.360 E-2	5.480 E-2	1.1375 E-6	1.4219 E-6
1.0	6.080 E-2	6.080 E-2	1.5897 E-6	1.5897 E-6

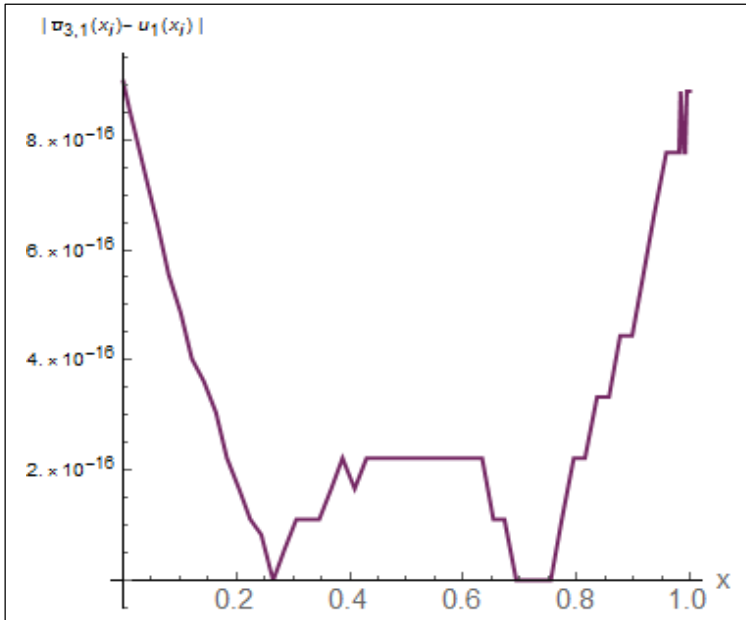


Figure 11: The absolute errors of the numerical solution of $\bar{u}_{3,1}(x)$ for $(n = 3)$. (Example 8.3)

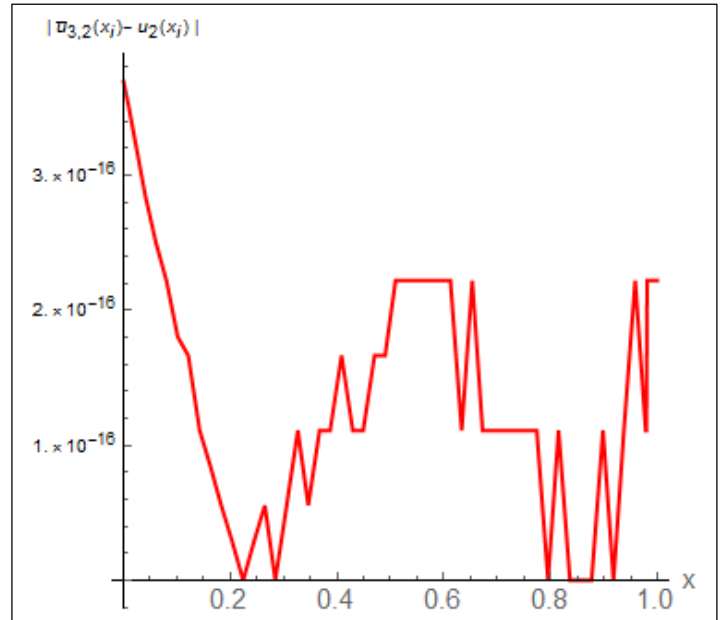


Figure 12: The absolute errors of the numerical solution of $\bar{u}_{3,2}(x)$ for $(n = 3)$. (Example 8.3)

9. Conclusion

In this paper, we presented a numerical method to solve system of linear fractional integro-differential equations. The numerical results showed the accuracy and effectiveness of the method compared with other methods such as homotopy perturbation method [7], Adomian decomposition method [8]. The graphs showed the convergence of the numerical solution to the analytical solution along the solution interval. We have computed the numerical results using the Mathematica 12 programming.

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