حلول تقريبية للمعادلات التفاضلية–التكاملية الكسرية باستخدام تحويل Kharrat-Toma مع طريقة الإضطراب الهوموتوبي

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□ ملخّص □

في هذه العمل، نقدم طريقة هجينة لحل المعادلات التفاضلية–التكاملية الكسرية الخطية (FIDEs)، مع تعريف المشتق الكسري وفق مفهوم كابوتو. ولأول مرة، يُستخدم هجين من تحويل Kharrat–Toma (KTT) وطريقة الإضطراب الهوموتوبي (HPM) بنجاح للحصول على حلول تقريبية لهذه الفئة من المعادلات. وقد تم إثبات فعالية التقنية المقترحة من خلال حل ثلاث مسائل توضيحية ومقارنة النتائج مع طرائق أخرى. أُجريت جميع الحسابات العددية وتحليلات الخطأ البيانية باستخدام برنامج ولفرام ماثيماتيكا.

الكلمات المفتاحية: المعادلات التفاضلية–التكاملية الكسرية، مشتق كابوتو الكسري، طريقة الإضطراب الهوموتوبي، تحويل Kharrat-Toma.

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Approximate Solutions of Fractional Integro-Differential Equations Using Kharrat-Toma Transform with Homotopy Perturbation Method

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□ Abstract □

In this paper, we present a hybrid method for solving linear fractional integro-differential equations (FIDEs), with the fractional derivative defined in the Caputo sense. For the first time, a hybrid of the Kharrat–Toma Transform (KTT) and the homotopy perturbation method (HPM) is successfully employed to obtain approximate solutions for this class of equations. The effectiveness of the proposed technique is demonstrated by solving three illustrative problems and comparing the results with other methods. All numerical computations and graphical error analyses were performed using the Wolfram Mathematica software.

Keywords: Fractional Integro-Differential Equations, Caputo Fractional Derivative, Homotopy Perturbation Method, Kharrat–Toma Transform.

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1. Introduction

The origins of fractional derivatives trace back to 1695, with Leibniz and L'Hospital being regarded as the first mathematicians to discuss the concept [1]. Fractional calculus finds extensive and diverse applications across engineering and scientific disciplines, including electrochemistry, electromagnetics, and signal processing [2], as well as in chaotic systems, nonlinear dynamical systems, heat transfer modeling, and the dynamics of interfaces between nanoparticles and substrates [3]. Fractional integro-differential equations (FIDEs) play a fundamental role in modeling numerous phenomena in science and engineering, including earthquake engineering, fluid mechanics, and biomedical engineering [4], along with control processes, electric circuit analysis, the dynamics of various systems, the activity of synaptically coupled networks of excitatory and inhibitory neurons, and viscoelastic material dynamics [5].

Nevertheless, many such models lack analytical or exact solutions. Consequently, researchers have recently developed a variety of numerical and analytical approaches to obtain solutions for this class of problems. For instance, Li [6] has used the generalized hat functions method to solve Volterra-Fredh

olm FIDEs. Hamoud and Ghadle [7] have used the Laplace decomposition method with Adomian decomposition method to solve Volterra-Fredholm FIDEs. Wang et al. [3] have used fractional-order Euler functions to solve FIDEs with a weakly singular kernel. Dascioglu and Bayram [8] have developed a collocation method based on Laguerre polynomials to solve Fredholm FIDEs. Sabeg et al. [9] have appled least squares method with aid of the Taylor expansion method to solve Volterra-Fredholm FIDEs. Saadatmandi et al. [1] have used a hybrid of Block-Pulse functions and Bernoulli polynomials to solve Volterra-Fredholm FIDEs. Henka et al. [10] have used

Hermite wavelets collocation method to solve Fredholm FIDEs. Taparki et al. [11] have used shifted Chebyshev polynomials and standard collocation points to solve Volterra FIDEs. Oyedepo et al. [12] have used Bernstein collocation technique points to solve Volterra FIDEs. Mohammadizadeh et al. [13] have used C^3-spline methods for solving FIDEs.

In this paper, we present a semi-analytical method for solving a fractional integro-differential equation of the type:

$$D^{\alpha}\varphi(x) = g(x) + A(x)\,\varphi(x) + \beta_1 \int_0^x k_1(x,t)\,\varphi(t)\,dt + \beta_2 \int_0^1 k_2(x,t)\,\varphi(t)\,dt\,,\ \ \alpha > 0 \qquad (1)$$

with the initial conditions

$$\varphi^{(i)}(0) = \varphi_0^{(i)}\,,\ \ i = 0,1,\dots,\lceil\alpha\rceil - 1\,. \qquad (2)$$

Here, $D^{\alpha}\varphi(x)$ denotes the α fractional derivative of $\varphi(x)$, with $0 \le x, t \le 1$. The functions $A, g: [0,1] \to \mathbb{R}$ *and* $k_i: [0,1]\times[0,1] \to \mathbb{R}$ (for $i = 1,2$) are known continuous. β_1 and β_2 are real constants, and $\varphi(x)$ is the unknown function to be determined.

2. Importance and Objectives of the Research:

This work focuses on obtaining approximate solutions for linear fractional integro-differential equations using a semi-analytical approach and evaluating the efficacy of the proposed method, given that these equations are pivotal in modeling a broad spectrum of phenomena in various scientific and engineering fields.

3. Methodology:

This research falls under the purview of applied mathematics, with particular emphasis on fractional integro-differential equations and numerical analysis. The proposed approach is based on the Kharrat-Toma transform and the homotopy perturbation method. Computational implementation and analysis were carried out using Mathematica 12.

4. Basic Definitions of Fractional Calculus:

In this section, we outline the main definitions of fractional calculus.

Definition 4.1 (see [14]). Riemann–Liouville Fractional Integral

The Riemann–Liouville fractional integral operator of order α for a function u is given by:

$$I^{\alpha}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\,\varphi(s)\,ds\,, \tag{3}$$

$where\ \ m-1<\alpha\leq m\ \ , m\in\mathbb{N}^*,\ \ and\ \Gamma(.)\ is\ the\ Gamma$ function.

Definition 4.2 (see [15]). Caputo's Fractional Derivative

The Caputo fractional derivative operator of order α for a function u is given by:

$$D^{\alpha}u(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-s)^{m-\alpha-1}\,u^{(m)}(s)\ ds \tag{4}$$

$where\ \ m-1<\alpha\leq m\,, m\in\mathbb{N}^*\,, x>0.$

Hence, we have the following properties [1,14]:

- $D^{\alpha}\ and\ I^{\alpha}\ are\ linear\ operators.$
- $D^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}\,x^{\gamma-\alpha}\quad ,\alpha>0\,,\gamma>-1\,,x>0.$

- $I^{\alpha}D^{\alpha}\varphi(x) = \varphi(x) - \sum_{k=0}^{m-1}\varphi^{(k)}(0^{+})\frac{x^{k}}{k!}$, $m-1 < \alpha < m.$
- $D^{\alpha}C = 0\,;\quad C\ is\ constant.$
- $D^{\alpha}x^{\gamma} = \begin{cases} 0 & ;\gamma\epsilon\mathbb{N}_0,\ \gamma < \lceil\alpha\rceil. \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\,x^{\gamma-\alpha} & ;\ \gamma\epsilon\mathbb{N}_0,\ \gamma \geq \lceil\alpha\rceil. \end{cases}$

5. Kharrat-Toma Transform

In 2020, Kharrat and Toma proposed a new integral transform.

Definition 5.1 [16]. Let $\varphi(x)$ denote a function on $[0, +\infty)$ that grows at most exponentially. The Kharrat–Toma transform (KTT) and its inverse (IKTT) are defined respectively by:

$$B[\varphi(x)] = G(S) = S^{3}\int_{0}^{\infty}\varphi(x)\,e^{\frac{-x}{s^{2}}}\,dx \quad , \qquad x \geq 0.$$

$$\varphi(x) = B^{-1}[G(S)] = B^{-1}\left[S^{3}\int_{0}^{\infty}\varphi(x)\,e^{\frac{-x}{s^{2}}}\,dx\right].$$

The KTT and IKTT for specific functions are presented below:

$(\mathbf{1})\ \varphi(x) = 1 \ \underset{B^{-1}}{\overset{B}{\rightleftharpoons}}\ G(S) = S^{5}$

$(\mathbf{1})\ \varphi(x) = x^{n} \ \underset{B^{-1}}{\overset{B}{\rightleftharpoons}}\ G(S) = S^{2n+5}\,.n!\,.$

$(\mathbf{3})\ \varphi(x) = \sin(a\,x) \ \underset{B^{-1}}{\overset{B}{\rightleftharpoons}}\ G(S) = \frac{a\,S^{7}}{1\ +\ a^{2}S^{4}}$

$(\mathbf{4})\ \varphi(x) = \cos(ax) \ \underset{B^{-1}}{\overset{B}{\rightleftharpoons}}\ G(S) = \frac{S^{5}}{1\ +\ a^{2}S^{4}}$

$(\mathbf{5})\ \varphi(x) = sinh(ax) \ \underset{B^{-1}}{\overset{B}{\rightleftharpoons}}\ G(S) = \frac{a\,S^{7}}{1\ -\ a^{2}S^{4}}$

$$(\mathbf{6})\ \varphi(x) = \cosh(ax) \ \underset{B^{-1}}{\overset{B}{\rightleftharpoons}}\ G(S) = \frac{S^5}{1 - a^2S^4}$$

Property 1: The KTT is a linear operator. If $a_1, a_2, ..., a_n$ are real constants, then:

$$B\left[\sum_{i=1}^{n} a_i\, \varphi_i(x)\right] = \left[\sum_{i=1}^{n} a_i\, B[\varphi_i(x)]\right].$$

Property 2: Let $M(S)$ and $N(S)$ denote the KTT of $\varphi(x)$ and $g(x)$ respectively. If

$$(\varphi * g)(x) = \int_0^x \varphi(\tau)\, g(x-\tau)\, d\tau\ , \tag{5}$$

where* denotes convolution of φ and g. Then, the KTT of the convolution of $\varphi(x)$ and $g(x)$ is:

$$B[\varphi(x) * g(x)] = \frac{1}{S^3}\, M(S)\, N(S).$$

Theorem 5.2 [17]. Let $m \in \mathbb{N}$ and $\alpha > 0$ such that $m - 1 < \alpha < m$. If $G(S)$ denotes the Kharrat- Toma Transform of the function $\varphi(x)$, then the Kharrat-Toma Transform of the Caputo fractional derivative of $\varphi(x)$ of order α is given by:

$$B[D^{\alpha}\varphi(x)] = S^{-2\alpha}\, G(S) - \sum_{k=0}^{m-1} S^{2k-2\alpha+5}\, \varphi^k(x)|_{x=0} \tag{6}$$

6. Mathematical Formulation of Proposed Method

In this section, we will explain the proposed hybrid KTT-HPM method. Considering Eq. (1), applying the KTT to both sides of Eq. (1) yields:

$$B[D^{\alpha}\varphi(x)] = B[g(x)] + B\big[A(x)\,\varphi(x)\big] + B\left[\beta_1 \int_0^x k_1(x,t)\,\varphi(t)\,dt + \beta_2 \int_0^1 k_2(x,t)\,\varphi(t)\,dt\right] \quad (7)$$

Using the differentiation property of KTT (6), we get

$$S^{-2\alpha}B[\varphi(x)] - \sum_{k=0}^{m-1} S^{2k-2\alpha+5}\,\varphi^k(0) = B[g(x)] + B[A(x)\,\varphi(x)] + B\left[\beta_1 \int_0^x k_1(x,t)\,\varphi(t)\,dt + \beta_2 \int_0^1 k_2(x,t)\,\varphi(t)\,dt\right] \quad (8)$$

From Eq. (8), we get:

$$B[\varphi(x)] = \sum_{k=0}^{m-1} S^{2k+5}\,\varphi^k(0) + S^{2\alpha}\,B[g(x)] + S^{2\alpha}\,B\left[A(x)\,\varphi(x) + \beta_1 \int_0^x k_1(x,t)\,\varphi(t)\,dt + \beta_2 \int_0^1 k_2(x,t)\,\varphi(t)\,dt\right] \quad (9)$$

Applying the IKTT, we get:

$$\varphi(x) = B^{-1}\left[\sum_{k=0}^{m-1} S^{2k+5}\,\varphi^k(0) + S^{2\alpha}\,B[g(x)]\right]$$

$$+B^{-1}\left[S^{2\alpha}B\left[A(x)\varphi(x)+\beta_1\int_0^x k_1(x,t)\,\varphi(t)dt\right.\right.$$

$$\left.\left.+\beta_2\int_0^1 k_2(x,t)\,\varphi(t)dt\right]\right] \qquad (10)$$

$$\varphi(x)=\psi(x)+B^{-1}\left[S^{2\alpha}B\left[A(x)\varphi(x)+\beta_1\int_0^x k_1(x,t)\,\varphi(t)\,dt\right.\right.$$

$$\left.\left.+\beta_2\int_0^1 k_2(x,t)\,\varphi(t)\,dt\right]\right] \qquad (11)$$

Here, the function $\psi(x)$ represents the term arising from both the nonhomogeneous part and the prescribed initial conditions.

The homotopy of equation (11) can be written as follows:

$$\varphi(x)=\psi(x)+P\,B^{-1}\left[S^{2\alpha}\,B\left[A(x)\,\varphi(x)+\beta_1\int_0^x k_1(x,t)\,\varphi(t)\,dt\right.\right.$$

$$\left.\left.+\beta_2\int_0^1 k_2(x,t)\,\varphi(t)\,dt\right]\right] \qquad (12)$$

where $P\in[0,1]$ is an embedding parameter. The function $\varphi(x)$ is represented as:

$$\varphi(x)=\sum_{i=0}^{\infty}P^i\varphi_i(x) \qquad (13)$$

Substituting equation (13) into equation (12), yields

$$\sum_{i=0}^{\infty} P^i\, \varphi_i(x) = \psi(x)$$

$$+ P\, B^{-1}\left[S^{2\alpha}\, B\left[A(x)\left(\sum_{i=0}^{\infty} P^i \varphi_i(x)\right) + \beta_1 \int_0^x k_1(x,t)\left(\sum_{i=0}^{\infty} P^i \varphi_i(t)\right) dt \right.\right.$$

$$\left.\left. + \beta_2 \int_0^1 k_2(x,t)\left(\sum_{i=0}^{\infty} P^i \varphi_i(t)\right) dt \right]\right] \qquad (14)$$

Comparing the coefficients of like powers of P in Eq. (14) yields:

$$P^0:\ \varphi_0 = \psi(x)\,,$$

$$P^1:\ \varphi_1 = B^{-1}\left[S^{2\alpha} B\left[A(x)\,\varphi_0(x) + \beta_1 \int_0^x k_1(x,t)\,\varphi_0(t)\,dt + \beta_2 \int_0^1 k_2(x,t)\,\varphi_0(t)\,dt \right]\right],$$

$$P^2:\ \varphi_2 = B^{-1}\left[S^{2\alpha} B\left[A(x)\,\varphi_1(x) + \beta_1 \int_0^x k_1(x,t)\,\varphi_1(t)\,dt + \beta_2 \int_0^1 k_2(x,t)\,\varphi_1(t)\,dt \right]\right],$$

$$\vdots$$

$$P^{n+1}:\ \varphi_{n+1} = B^{-1}\left[S^{2\alpha} B\left[A(x)\,\varphi_n(x) + \beta_1 \int_0^x k_1(x,t)\,\varphi_n(t)\,dt + \beta_2 \int_0^1 k_2(x,t)\,\varphi_n(t)\,dt \right]\right],$$

where $n = 0,1,2, \ldots$

Finally, this gives the functions φ_n for $n = 0,1,2, \ldots$. Setting $P = 1$, we have the approximate solution of problem (1)-(2) as follows:

$$\bar{\varphi}(x) = \sum_{i=0}^{n} \varphi_i (x). \tag{15}$$

7. Illustrative Examples

In this section, we test the effectiveness of the proposed method by solving three illustrative examples. All numerical results and graphical representations of solutions and corresponding errors were generated using Wolfram Mathematica. The absolute error is denoted as follows:

$$Absolute\ Error = |\varphi(x) - \bar{\varphi}(x)| \ ; \ 0 \le x \le 1$$

where $\varphi(x)$ is the exact solution and $\bar{\varphi}(x)$ is the approximate solution.

Example 7.1. ([1],[6]): Consider the following fractional Integro-differential equation:

$$D^{2.3}\varphi(x) = \frac{\Gamma(4.5)}{\Gamma(2.2)}\, x^{1.2} - \frac{1}{99}\, x^{5.5} - \frac{1}{11}\, x + \frac{1}{4}\int_0^x (x-t)\, \varphi(t)\, dt + \frac{1}{2}\int_0^1 x\, t\, \varphi(t) dt, \tag{16}$$

with the initial conditions $\varphi(0) = \varphi`(0) = \varphi``(0) = 0$, the exact solution is $\varphi(x) = x^{3.5}$.

Solution: Applying the KTT to both sides of Eq. (16) yields:

$$B[D^{2.3}\varphi(x)] = B\left[\frac{\Gamma(4.5)}{\Gamma(2.2)}\, x^{1.2} - \frac{1}{99}\, x^{5.5} - \frac{1}{11}\, x\right] + B\left[\frac{1}{4}\int_0^x (x-t)\, \varphi(t)\, dt + \frac{1}{2}\int_0^1 x\, t\, \varphi(t)\, dt\right] \tag{17}$$

Using the differentiation property of KTT (6), we get

$$S^{-4.6}B[\varphi(x)] = B\left[\frac{\Gamma(4.5)}{\Gamma(2.2)}\,x^{1.2} - \frac{1}{99}\,x^{5.5} - \frac{1}{11}\,x\right] + B\left[\frac{1}{4}\int_0^x (x-t)\,\varphi(t)\,dt + \frac{1}{2}\int_0^1 x\,t\,\varphi(t)\,dt\right] \quad (18)$$

From Eq. (18), we get:

$$B[\varphi(x)] = S^{4.6}\,B\left[\frac{\Gamma(4.5)}{\Gamma(2.2)}\,x^{1.2} - \frac{1}{99}\,x^{5.5} - \frac{1}{11}\,x\right] + S^{4.6}\,B\left[\frac{1}{4}\int_0^x (x-t)\,\varphi(t)\,dt + \frac{1}{2}\int_0^1 x\,t\,\varphi(t)\,dt\right] \quad (19)$$

Applying the IKTT, we get:

$$\varphi(x) = B^{-1}\left[S^{4.6}\,B\left[\frac{\Gamma(4.5)}{\Gamma(2.2)}\,x^{1.2} - \frac{1}{99}\,x^{5.5} - \frac{1}{11}\,x\right]\right] + B^{-1}\left[S^{4.6}B\left[\frac{1}{4}\int_0^x (x-t)\,\varphi(t)dt + \frac{1}{2}\int_0^1 x\,t\,\varphi(t)\,dt\right]\right] \quad (20)$$

The homotopy of equation (20) can be written as follows:

$$\varphi(x) = B^{-1}\left[S^{4.6}\,B\left[\frac{\Gamma(4.5)}{\Gamma(2.2)}\,x^{1.2} - \frac{1}{99}\,x^{5.5} - \frac{1}{11}\,x\right]\right] + P\,B^{-1}\left[S^{4.6}B\left[\frac{1}{4}\int_0^x (x-t)\,\varphi(t)\,dt + \frac{1}{2}\int_0^1 x\,t\,\varphi(t)\,dt\right]\right] \quad (21)$$

where $P \in [0,1]$. The function $\varphi(x)$ is represented as:

$$\varphi(x) = \sum_{i=0}^{\infty} P^i \varphi_i(x) \tag{22}$$

Substituting equation (22) into equation (21), yields

$$\varphi(x) = B^{-1}\left[S^{4.6}\, B\left[\frac{\Gamma(4.5)}{\Gamma(2.2)}\, x^{1.2} - \frac{1}{99}\, x^{5.5} - \frac{1}{11}\, x\right]\right] + p\, B^{-1}\left[S^{4.6}\, B\left[\frac{1}{4}\int_0^x (x-t)\left(\sum_{i=0}^{\infty} P^i \varphi_i(x)\right) dt + \frac{1}{2}\int_0^1 x\, t\left(\sum_{i=0}^{\infty} P^i \varphi_i(x)\right) dt\right]\right] \tag{23}$$

Comparing the coefficients of like powers of P in Eq. (23) yields:

$$P^0:\ \varphi_0(x) = B^{-1}\left[S^{4.6}\, B\left[\frac{\Gamma(4.5)}{\Gamma(2.2)}\, x^{1.2} - \frac{1}{99}\, x^{5.5} - \frac{1}{11}\, x\right]\right]$$

$$\varphi_0(x) = B^{-1}[-2.907932099141862\ S^{20.6} + 11.631728396567448\ S^{12} - 0.09090909090909091\, S^{11.6}]$$

$$= -0.01026601535464682\, x^{3.3} + 0.9999999999999999\, x^{3.5} - 0.0001103999090965403\, x^{7.8}.$$

$$P^1:\ \varphi_1(x) = B^{-1}\left[S^{4.6}\, B\left[\frac{1}{4}\int_0^x (x-t)\, \varphi_0(t)\, dt + \frac{1}{2}\int_0^1 x\, t\, \varphi_0(t)\, dt\right]\right],$$

$$\varphi_1(x) = B^{-1}[-0.7269830247854662\ S^{29.2} + 2.9079320991418607\ S^{12} - 0.022727272727272717\ S^{11.6} + 0.08993496624607333\ S^{11.6}]$$

$$= 0.010156011188420156\ x^{3.3} - 0.000001314456105974719\ x^{7.6} + 0.00011039990909654025\ x^{7.8} - 1.178451115365009 \times 10^{-9}\ x^{12.1}.$$

$$P^2:\ \varphi_2(x) = B^{-1}\left[S^{4.6}\ B\left[\frac{1}{4}\int_0^x (x-t)\ \varphi_1(t)\ dt + \frac{1}{2}\int_0^1 x\ t\ \varphi_1(t)\ dt\right]\right],$$

$$\varphi_2(x) = B^{-1}[-0.18174575619636635\ S^{29.2} + 0.7269830247854652\ S^{12} - 0.005681818181818175\ S^{11.6} + 0.02248374156151833\ S^{11.6}]$$

$$= 0.00010882451071753478\ x^{3.3} + 0.000001300371220750595\ x^{7.6} - 1.526436871356955 \times 10^{-11}\ x^{11.9} + 1.178451115365008 \times 10^{-9}\ x^{12.1} - 2.816672204590534 \times 10^{-15}\ x^{16.4}.$$

Setting $P = 1$, the approximate solution to Problem (16) for the first three terms of the solution series (22) is:

$$\bar{\varphi}(x) = \varphi_0(x) + \varphi_1(x) + \varphi_2(x)\ ,$$

$$\bar{\varphi}(x) = -\ 0.000001179655509130073\ x^{3.3} + 0.9999999999999999\ x^{3.5}$$

$$- 1.408488522412398 \times 10^{-8}x^{7.6} - 5.421010862427522 \times 10^{-20}x^{7.8}$$
$$- 1.526436871356955 \times 10^{-11}x^{11.9} - 1.654361225106055 \times 10^{-24}x^{12.1}$$
$$- 2.816672204590534 \times 10^{-15}\, x^{16.4}. \qquad (24)$$

Table 1. The Numerical Results of Example 7.1.

		Proposed Method (KTT-HPM) (n = 2)	
x	Exact Solution	Approximate Solution	Absolute Error
$1/8$	0.00069053397	0.00069053273130967	1.23469 E-9
$2/8$	0.0078125	0.007812487838963489	1.21610 E-8
$3/8$	0.0322930777	0.03229303130456822	4.63591 E-8
$4/8$	0.0883883476	0.08838822780346454	1.19845 E-7
$5/8$	0.193010111	0.19300986057219424	2.50522 E-7
$6/8$	0.365354467	0.36535400912158106	4.58099 E-7
$7/8$	0.626654533	0.6266537686857518	7.64353 E-7

Table 1 shows the numerical results obtained by our method for Example 7.1 with n = 2. Table 2 shows the absolute errors of existing techniques, such as generalized hat functions, CAS wavelets [6], Legendre wavelets, and a hybrid method [1]. A comparison between the two tables reveals that the proposed method achieves smaller absolute errors than the approaches in [1, 6]. It is worth emphasizing that this high accuracy is attained with only n = 2, whereas the competing methods require n = 32, underscoring the computational efficiency of our method.

	Absolute errors of Li Method [6]		**Absolute errors of Saadatmandi Method [1]**		
x	**Hat**	**CASW**	**LWM**	**Hybrid Method**	
	$(n = 32)$	$(n = 32)$	$(m = 2, k = 5)$	$(m = 6, n = 2)$	$(m = 7, n = 2)$

$1/8$	4.2465 E-6	1.0174 E-4	6.64 E-5	9.89 E-6	6.76 E-6
$2/8$	5.6172 E-5	4.1987 E-4	4.53 E-5	2.47 E-5	1.69 E-5
$3/8$	6.2164 E-5	9.2364 E-4	3.14 E-5	4.21 E-5	2.89 E-5
$4/8$	6.9423 E-5	4.1726 E-3	7.37 E-5	6.15 E-5	4.20 E-5
$5/8$	3.2375 E-4	8.1648 E-3	2.44 E-4	8.16 E-5	5.70 E-5
$6/8$	4.5421 E-4	2.3112 E-2	3.81 E-4	1.03 E-4	7.30 E-5
$7/8$	6.2376 E-4	8.0723 E-2	6.02 E-4	1.25 E-4	9.02 E-5

Table .2. describes the absolute errors obtained in ([1],[6]).

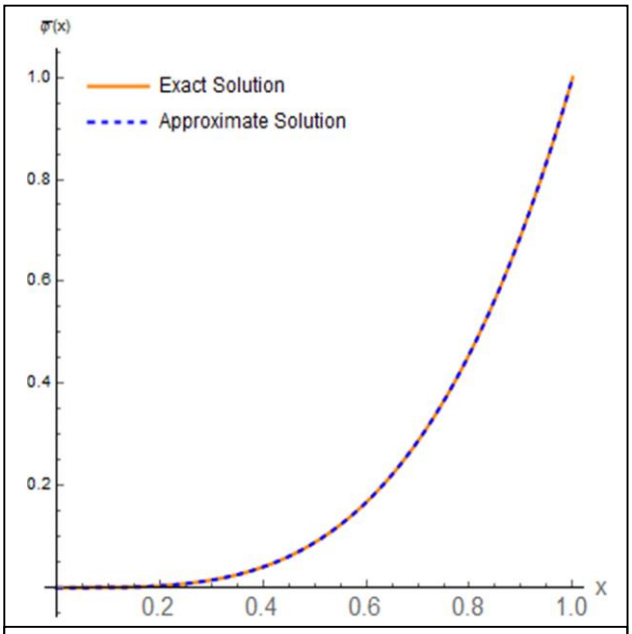


Figure 1: Exact and approximate solutions of Example 7.1 with (n = 2).

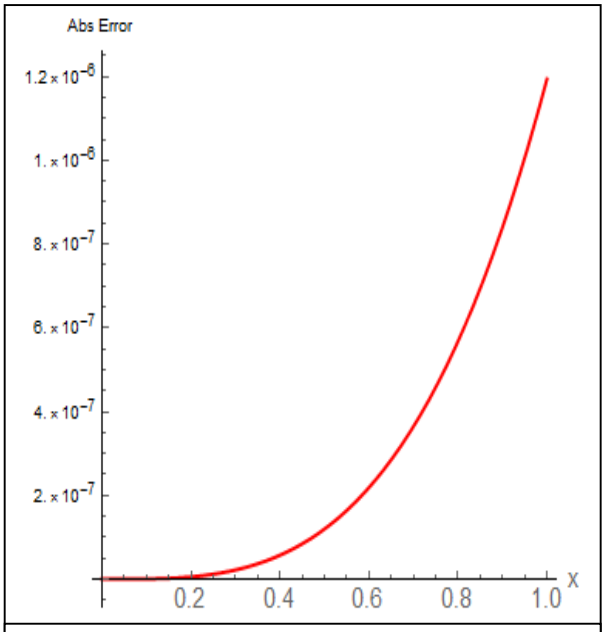


Figure 2: The absolute errors of Example 7.1 with (n = 2).

Example 7.2. [18]: Consider the following FIDE:

$$D^{1/2}\varphi(x) = \frac{\left(\frac{8}{3}\ x^{\frac{3}{2}} - 2\,x^{\frac{1}{2}}\right)}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 (x\,t)\ \varphi(t)\,dt\ \ 0 \le x \le 1 \qquad (25)$$

with the initial condition $\varphi(0) = 0$, the exact solution is $\varphi(x) = x^2 - x$.

Solution: Applying the KTT to both sides of Eq. (25) yields:

$$B\left[D^{1/2}\varphi(x)\right] = B\left[\frac{(\frac{8}{3}\ x^{\frac{3}{2}} - 2\,x^{\frac{1}{2}})}{\sqrt{\pi}} + \frac{x}{12}\right] + B\left[\int_0^1 (x\,t)\ \varphi(t)\,dt\right] \qquad (26)$$

Using the differentiation property of KTT (6), we get

$$S^{-1}\,B[\varphi(x)] = B\left[\frac{(\frac{8}{3}\ x^{\frac{3}{2}} - 2\,x^{\frac{1}{2}})}{\sqrt{\pi}} + \frac{x}{12}\right] + B\left[\int_0^1 (x\,t)\ \varphi(t)\,dt\right] \quad (27)$$

From Eq. (27), we get:

$$B[\varphi(x)] = S\,B\left[\frac{(\frac{8}{3}\ x^{\frac{3}{2}} - 2\,x^{\frac{1}{2}})}{\sqrt{\pi}} + \frac{x}{12}\right] + S\,B\left[\int_0^1 (x\,t)\ \varphi(t)\,dt\right] \quad (28)$$

Applying the IKTT, we get:

$$\varphi(x) = B^{-1}\left[S\,B\left[\frac{(\frac{8}{3}\ x^{\frac{3}{2}} - 2\,x^{\frac{1}{2}})}{\sqrt{\pi}} + \frac{x}{12}\right]\right] + B^{-1}\left[S\,B\left[\int_0^1 (x\,t)\ \varphi(t)\,dt\right]\right] \quad (29)$$

The homotopy of equation (29) can be written as follows:

$$\varphi(x) = B^{-1}\left[S\,B\left[\frac{(\frac{8}{3}\ x^{\frac{3}{2}} - 2\,x^{\frac{1}{2}})}{\sqrt{\pi}} + \frac{x}{12}\right]\right] + P\,B^{-1}\left[S\,B\left[\int_0^1 x\,t\,\varphi(t)dt\right]\right] \quad (30)$$

where $P \in [0,1]$. The function $\varphi(x)$ is represented as:

$$\varphi(x) = \sum_{i=0}^{\infty} P^i \varphi_i(x) \qquad (31)$$

Substituting equation (31) into equation (30), yields

$$\varphi(x) = B^{-1}\left[S\,B\left[\frac{\left(\frac{8}{3}\,x^{\frac{3}{2}} - 2\,x^{\frac{1}{2}}\right)}{\sqrt{\pi}} + \frac{x}{12}\right]\right] + P\,B^{-1}\left[S\,B\left[\int_0^1 x\,t\left(\sum_{i=0}^{\infty} P^i \varphi_i(x)\right)dt\right]\right] \quad (32)$$

Comparing the coefficients of like powers of P in Eq. (32) yields:

$$P^0:\ \varphi_0(x) = B^{-1}\left[S\,B\left[\frac{(\frac{8}{3}\,x^{\frac{3}{2}} - 2\,x^{\frac{1}{2}})}{\sqrt{\pi}} + \frac{x}{12}\right]\right],$$

$$\varphi_0(x) = B^{-1}[1.9999999999999998\ S^9 - 0.9999999999999999\ S^7 + 0.08333333333333331\ S^8]$$

$$= -0.9999999999999999\,x + 0.06268773150530625\,x^{1.5} + 0.9999999999999999\,x^2.$$

$$P^1:\ \varphi_1(x) = B^{-1}\left[S\,B\ \left[\int_0^1 x\,t\,\varphi_0(x)\,dt\right]\right]$$

$$= B^{-1}[-0.06542255290324642\,S^8] = -0.04921429716948487\,x^{1.5}.$$

$$P^2:\ \varphi_2(x) = B^{-1}\left[S\,B\ \left[\int_0^1 x\,t\,\varphi_1(x)\,dt\right]\right]$$

$$= B^{-1}[-0.01406122776270947\,S^8] = -0.010577597647484272\,x^{1.5}.$$

$$P^3:\ \varphi_3(x) = B^{-1}\left[S\,B\ \left[\int_0^1 x\,t\,\varphi_2(x)\,dt\right]\right]$$

$$= B^{-1}[-0.003022170756423972\ S^8] = -0.002273436347302731\ x^{1.5}.$$

$$P^4: \varphi_4(x) = B^{-1}\left[S\ B\left[\int_0^1 x\ t\ \varphi_3(x)dt\right]\right]$$

$$= B^{-1}[-0.000649553242086472\ S^8] = -0.0004886282308598153\ x^{1.5}.$$

Setting $P = 1$, the approximate solution to Problem (25) for the first five terms of the solution series (31) is:

$$\bar{\varphi}(x) = \varphi_0(x) + \varphi_1(x) + \varphi_2(x) + \varphi_3(x) + \varphi_4(x)\ ,$$

$$\bar{\varphi}(x) = -0.9999999999999999\ x + 0.00013377211017455426\ x^{1.5} + 0.9999999999999999\ x^2\ . \quad (33)$$

Table 3. displays the absolute errors for Example 7.2, along with a comparison to the absolute errors obtained in [18].

		Proposed Method (KTT-HPM) (n = 4)		**LSM-Chebyshev [18] (n = 5)**
x	**Exact Solution**	**Approximate Solution**	**Absolute Error**	**Absolute Error**
0.0	0.00	0.00	0.00	4.1207 E−3
0.1	−0.09	−0.08999576975444441	4.23025 E-6	3.2738 E-3
0.2	−0.16	−0.15998803505872622	1.19649 E-5	2.4127 E-3
0.3	−0.21	−0.20997801899930768	2.19810 E-5	1.5405 E-3
0.4	−0.24	−0.2399661580355553	3.38420 E-5	6.5928 E-4
0.5	−0.25	−0.24995270441688092	4.72956 E-5	2.2948 E-4
0.6	−0.24	−0.2399378283414128	6.21717 E-5	1.1250 E-3
0.7	−0.21	−0.20992165475592628	7.83452 E-5	2.0267 E-3
0.8	−0.16	−0.15990428046980987	9.57195 E-5	2.9344 E-3
0.9	−0.09	−0.0898857833699992	1.14217 E-4	3.8476 E-3
1.0	0.00	0.000133772110174513	1.33772 E-4	4.7658 E-3

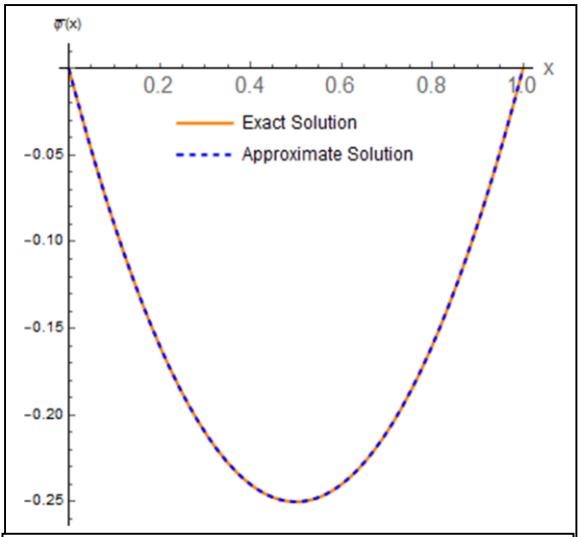


Figure 3: Exact and approximate solutions of Example 7.2 with (n = 4).

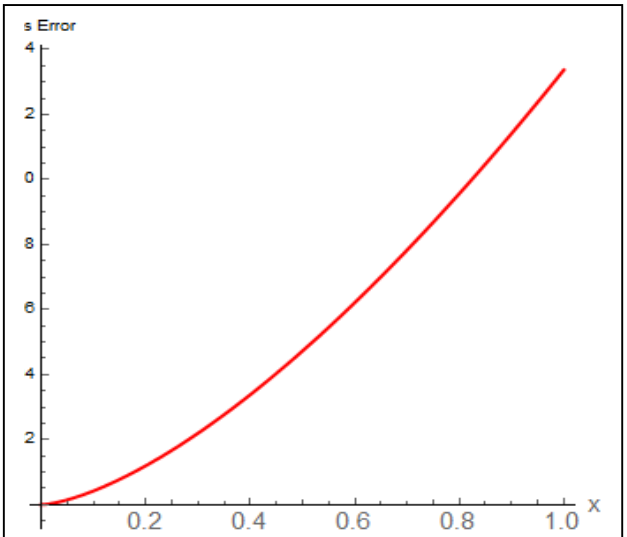


Figure 4: The absolute errors of Example 7.2 with ($n = 4$).

Example 7.3 [19]: Consider the following FIDE:

$$D^{1/3}\varphi(x) = \frac{3\sqrt{\pi}}{4\Gamma\left(\frac{13}{6}\right)} x^{\frac{7}{6}} - \frac{2}{63} x^{\frac{9}{2}} (9 + 7x^2) + \int_0^x (x\tau + x^2\tau^2)\, \varphi(\tau)\, d\tau \qquad (34)$$

with the initial condition $\varphi(0) = 0$, the exact solution is $\varphi(x) = x^{\frac{3}{2}}$.

Solution: Applying the proposed KTT-HPM method, we obtain the approximate solution to Problem (34) for the first three terms as:

$$\bar{\varphi}(x) = 0.9999997609267423\, x^{1.50000034} - 0.1651944752465806\, x^{29/6} - 0.11523321931834647\, x^{41/6} + 0.16519441609185467\, x^{4.833333673333334}$$

$$+ 0.11523318124297714\ x^{6.833333673333334} - 0.01184356816495771\ x^{8.166666666666666} - 0.015586236351191099\ x^{10.166666666666666} - 0.005049007138437858\ x^{12.166666666666666} + 0.0118435631768349\ x^{8.166667006666668} + 0.01558623011936537\ x^{10.166667006666668} - 0.0005111676589319269\ x^{11.5} + 0.005049005249812732\ x^{12.166667006666668} - 0.0009754170425105203\ x^{13.5} - 0.000613321948875309\ x^{15.499999999999996} - 0.00012741527133361416\ x^{17.499999999999996}\ . \quad (35)$$

Table 4. displays the absolute errors obtained in [20].

	Absolute Error Kumar Method [20] $(n = 10)$		
x	**Linear scheme** S_1	**Quadratic scheme** S_2	**Linear-Quadr scheme** S_3
0.2	9.7603 E-3	9.7603 E-3	9.7603 E-3
0.4	1.0859 E-2	4.8028 E-3	4.8668 E-3
0.6	1.1849 E-2	2.8790 E-3	3.1840 E-3
0.8	1.4118 E-2	2.6372 E-3	3.5232 E-3

Table 4 shows the absolute errors obtained by three existing numerical schemes [20]. Table 5 shows the numerical results for Example 7.3 using our method (with n = 2) alongside the absolute errors from the methods in [11,19] (with n = 3). Comparing the two tables demonstrates that our method yields smaller absolute errors than the methods in [11,19,20]. It is noteworthy that this enhanced accuracy is achieved with only n = 2, whereas the competing schemes require n = 10 and the methods in [11,19] require n = 3, thereby confirming the computational efficiency of our method.

Table 5. shows the absolute errors for Example 7.3, along with a comparison to the absolute errors obtained in [11, 19].

		Proposed Method (KTT-HPM) $(\mathbf{n = 2})$		**Method [11]** $(\mathbf{n = 3})$	**Method [19]** $(\mathbf{n = 3})$
x	**Exact solution**	**Approximate solution**	**Absolute Error**	**Absolute Error**	**Absolute Error**
0.1	0.0316227766	0.0316227442819465	3.23197 E-8		

0.2	0.0894427191	0.0894426487034592	7.03965 E-8	3.8288 E-3	2.9 E-4
0.3	0.164316767	0.1643166597186795	1.07533 E-7		
0.4	0.2529822128	0.2529820539215242	1.58892 E-7	1.1116 E-3	7.3 E-4
0.5	0.353553391	0.3535529440907343	4.46503 E-7		
0.6	0.4647580015	0.4647551369643616	2.86458 E-6	8.9670 E-4	9.5 E-4
0.7	0.585662019	0.585642740556613	1.92780 E-5		
0.8	0.715542	0.715432387371549	1.09365 E-4	1.7903 E-3	8.3 E-4
0.9	0.8538149682	0.8532873197674683	5.27648 E-4		

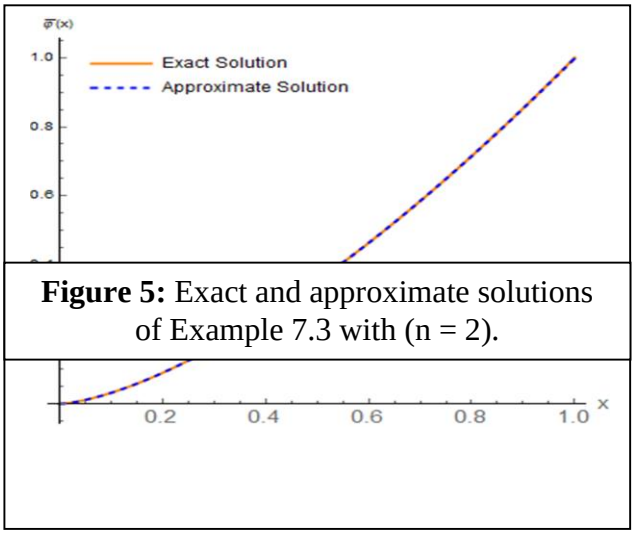


Figure 5: Exact and approximate solutions of Example 7.3 with (n = 2).

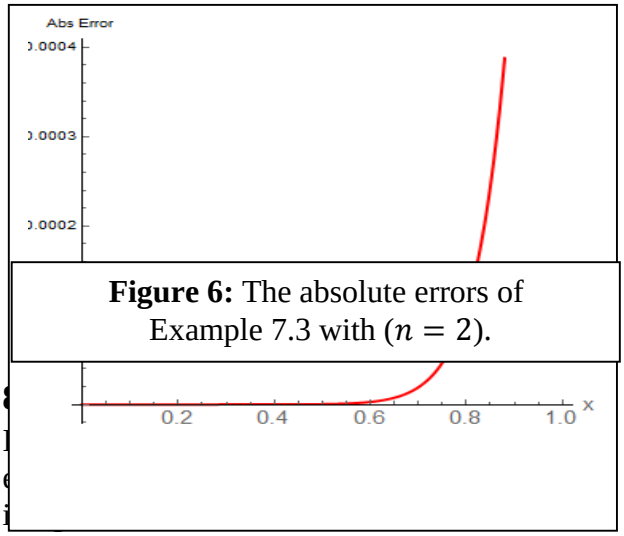


Figure 6: The absolute errors of Example 7.3 with ($n = 2$).

transform with the homotopy perturbation method to solve linear fractional integro-differential equations (FIDEs). The proposed method demonstrates higher accuracy compared to several existing numerical methods, including the hybrid of Block-Pulse functions and Bernoulli polynomials, Legendre wavelets method [1], the generalized hat functions method, CAS wavelet method [6], the least squares method with the aid of shifted Chebyshev polynomials [18], the collocation method based on Laguerre polynomials [19], the collocation method based on shifted Chebyshev polynomials [11], and three other numerical shemes [20].

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